ME 451B Flow Instabilities

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Topics to be covered:

1. Laminar-stability theory: Laminar-turbulent transition
2. Dynamical systems
   - Bifurcations
   - Global stability, linear stability of parallel flows, conditions for stability
   - Viscosity as a destabilizing factor, convective and absolute instability
3. Rayleigh equation
   - Instability criteria, response to small inviscid disturbances
4. Discussion of a selection of instabilities
   - Kelvin-Helmholtz, Rayleigh-Taylor, Richtmyer-Meshkov
   - Other instabilities in geophysical flows
5. The Orr-Sommerfeld equation
   - Dual role of viscosity, boundary-layer stability
6. Modern concepts
   - Transient growth
   - Non-normal character of the linear Navier-Stokes operator
   - Weakly nonlinear stability theory, phenomenological theories of turbulence

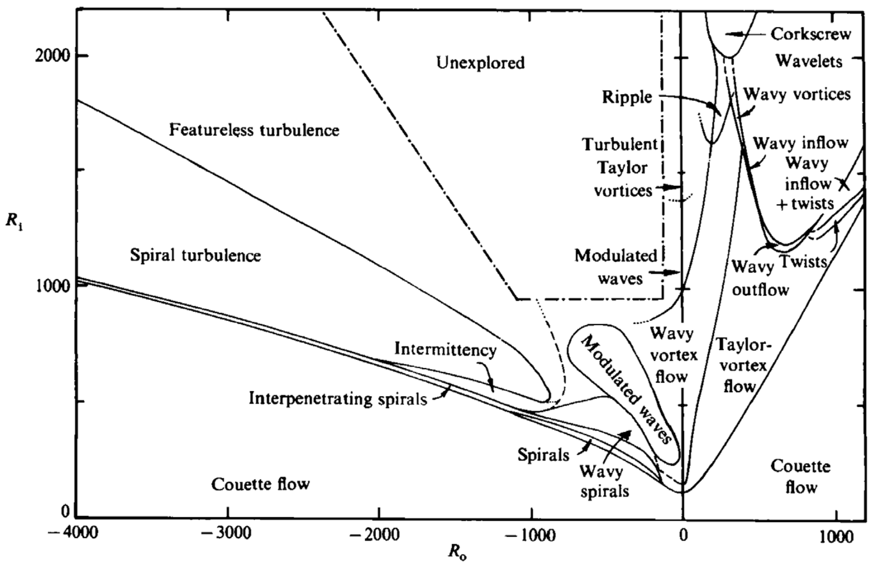
Textbooks:

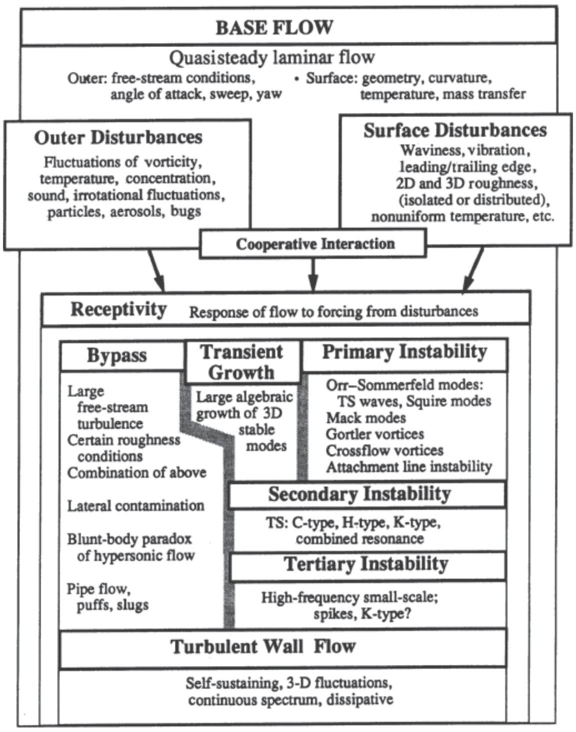
- Drazin & Reid, *Hydrodynamic Stability*
- Schmid & Henningson, *Stability and Transition in Shear Flows*

Introduction to Hydrodynamic Stability

➢ Transition from laminar to turbulence

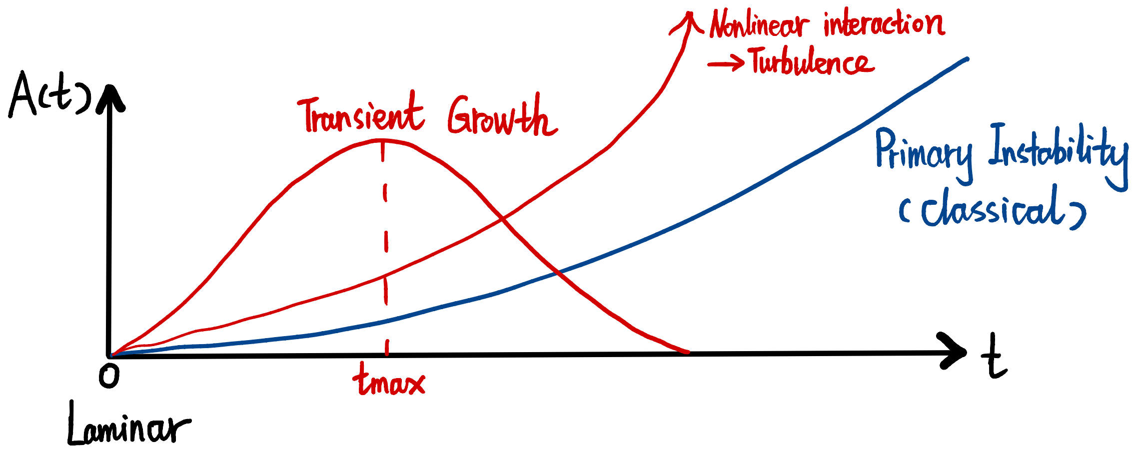
Regimes observed in flow between independently rotating concentric cylinders: From laminar Couette flows to turbulent Taylor vortices.





Laminar-turbulent transition (e.g., Morkovin map of the roads to wall turbulence)

- ♦ Primary instability and eigen-analysis: Exponential growing modes, infinite time
- ♦ Transient growth: Eventually decay after reaching the peak amplitude, finite time
- ♦ Bypass transition: Large free stream forcing, pipe flow

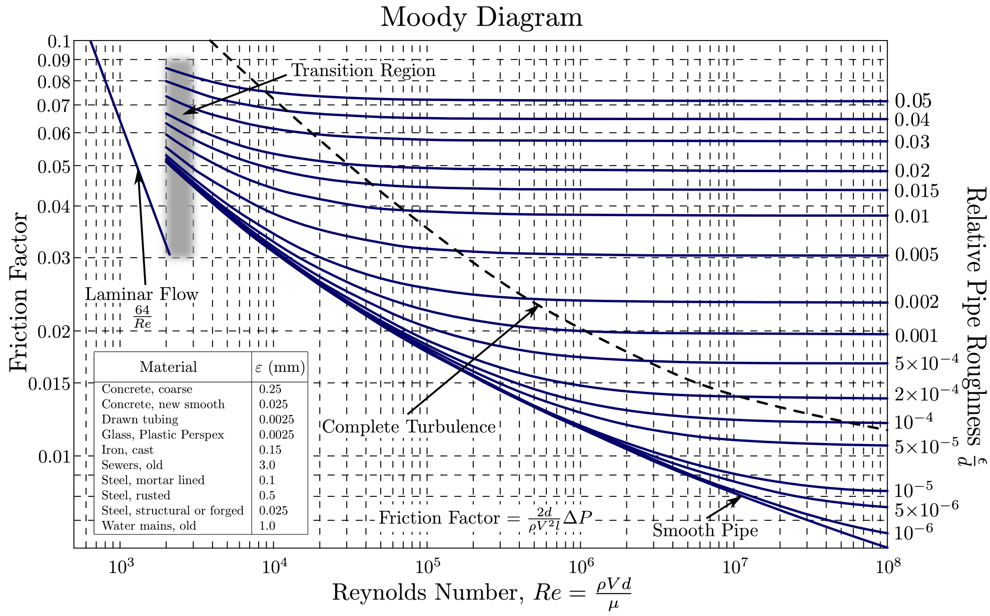


Two fundamental hypotheses about viscosity

- The inviscid fluid flow may be unstable and the viscous fluid flow is stable. The effect of viscosity is stabilizing.
- The inviscid fluid may be stable and the viscous fluid flow is unstable. The effect of viscosity is destabilizing.

Effect of viscosity

- The effect of viscosity is to dissipate energy and therefore it can damp out disturbances, stabilizing any flow for large enough viscosity.
- However, viscosity also diffuses momentum, which can destabilize some flows. In parallel shear flows, they are inviscidly (linearly) stable.



Material	ε (mm)
Concrete, coarse	0.25
Concrete, new smooth	0.025
Drawn tubing	0.0025
Glass, Plastic Perspex	0.0025
Iron, cast	0.15
Sewers, old	3.0
Steel, mortar lined	0.1
Steel, rusted	0.5
Steel, structural or forged	0.025
Water mains, old	1.0

- Hydrodynamic stability theory
- Response of a laminar flow to a disturbance of small or moderate amplitude.
- Study under what conditions a small disturbance to a laminar base flow either grows or decays. The state of the system can become a different laminar state, turbulent, or etc.
- Obtain the mathematical analysis of the evolution of disturbances while they are small. Linear governing equations are desirable, while open questions exist for solutions to the linear equations once nonlinearity is involved (i.e., disturbance amplitudes exceeds a few percent of the base flow).
- Experiments and simulations are difficult by nature, since disturbances are small to begin with, and environments (or errors and noise) must be controlled to avoid contamination.

- Approaches to tackling stability problems

Observations

Experiments clearly show transition, but the capture of onset of instability is difficult by nature. However, capturing the linear growth of perturbation is possible.

Classical methods

- Normal mode / Eigenvalue analysis: Perturbation growth on an infinite time horizon
- Initial value analysis

Modern methods

- Non-normal mode analysis / Transient growth: Growth on a finite time horizon, maybe followed by decay if transition does not occur
- Energy, phase-space methods, etc.

Basic idea: Classical modal solution technique

Originally developed for the stability of particles or structures, adopted to fluid flows.

Complications

- Simpler analysis in modal settings, but real problems may be initial value (IV) ones.
- Modal approach may fail (e.g., in pipe flows). Need non-normal approach.
- Base flows may be spatially or temporally evolving.
- Boundary conditions may be important (e.g., bounded and unbounded flows)

Linear Stability Analysis

- Governing equations

Incompressible flow of Newtonian fluid. The base flow has $\boldsymbol{u}(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t)$. The perturbed solutions are $\tilde{\boldsymbol{u}}(\boldsymbol{x},t)$ and $\tilde{p}(\boldsymbol{x},t)$. Both satisfy the equations under given boundary conditions.

Momentum (Navier-Stokes) & Continuity

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 \tilde{u}_i, \qquad \frac{\partial \tilde{u}_i}{\partial x_i} = 0$$

Boundary conditions

Typically, on solid boundaries, we have

$$\tilde{u}_i(x_i, 0) = \tilde{u}_i(x_i), \qquad \tilde{u}_i(x_i, t) = 0$$

Perturbation (Reynolds decomposition)

$$\tilde{u}_i = U_i + u_i, \qquad \tilde{p} = P + p$$

Base flow equation

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 U_i, \qquad \frac{\partial U_i}{\partial x_i} = 0$$

Usually we assume a time-independent base flow, so no evolution term.

Perturbation equation

We start from the full perturbed solutions

$$\frac{\partial (U_i + u_i)}{\partial t} + \left(U_j + u_j\right) \frac{\partial (U_i + u_i)}{\partial x_j} = -\frac{\partial (P + p)}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 (U_i + u_i), \qquad \frac{\partial (U_i + u_i)}{\partial x_i} = 0$$

Subtracting the base flow equation, we have

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}, \qquad \frac{\partial u_i}{\partial x_i} = 0$$

Linearized equation

For small perturbation amplitude, we neglect the second order term.

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i, \qquad \frac{\partial u_i}{\partial x_i} = 0$$

Disturbance size and growth defined from kinetic energy

The kinetic energy per volume, for an incompressible and isothermal flow, is

$$E_v = \frac{1}{2}\int_V u_i u_i \, \mathrm{d}V$$

The volume can be chosen as the infinite half-space for the boundary layer, or one wavelength of disturbance for a channel, etc.

The disturbance energy evolves as governed by the Reynolds-Orr equation. The perturbation grows from the balance between extraction of energy from the base flow and energy dissipation due to viscous effects.

- Definitions of stability

1. **General stability**

From ODE, the base flow is stable in the Lyapunov sense if for any $\varepsilon > 0$, $\exists\delta(\varepsilon) > 0$ such that

$$\|\tilde{\boldsymbol{u}}(\boldsymbol{x},0) - \boldsymbol{U}(\boldsymbol{x},0)\| < \delta \qquad \Longrightarrow \qquad \|\tilde{\boldsymbol{u}}(\boldsymbol{x},t) - \boldsymbol{U}(\boldsymbol{x},t)\| < \varepsilon, \qquad \forall t \geq 0$$

2. **Asymptotic stability**

$$\|\tilde{\boldsymbol{u}}(\boldsymbol{x},t) - \boldsymbol{U}(\boldsymbol{x},t)\| \to 0, \qquad t \to \infty$$

3. **Strong stability**

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\boldsymbol{u}}(\boldsymbol{x},t) - \boldsymbol{U}(\boldsymbol{x},t)\| < 0, \qquad \forall\tilde{\boldsymbol{u}}(\boldsymbol{x},0), \quad \forall t > 0$$

4. **Stability**

$$\lim_{t\to\infty}\frac{E_v(t)}{E_v(0)} \to 0$$

5. **Conditional stability**

If there exists a threshold energy $\delta > 0$ such that the flow is stable if $E(0) < \delta$.

6. **Global stability**

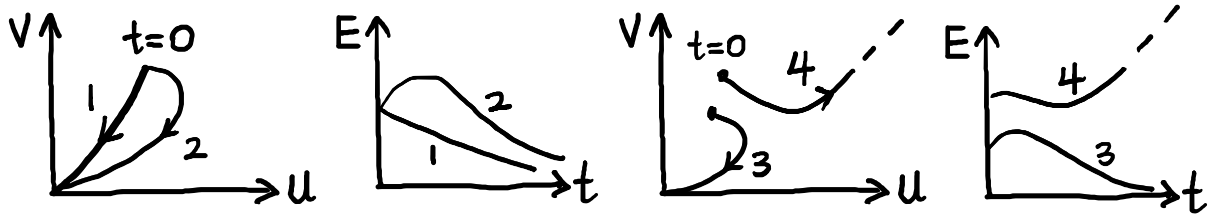
If the threshold energy $\delta \to \infty$. This is a special case of conditional stability.

7. **Monotonic stability**

$$\frac{\mathrm{d}E_v}{\mathrm{d}t} < 0, \qquad \forall t > 0$$

Example of perturbation growth

Path 1 has monotonic stability to the initial condition, while path 2 has non-monotonic stability. Paths 3 and 4 indicate conditional stability, since there exists a threshold energy.

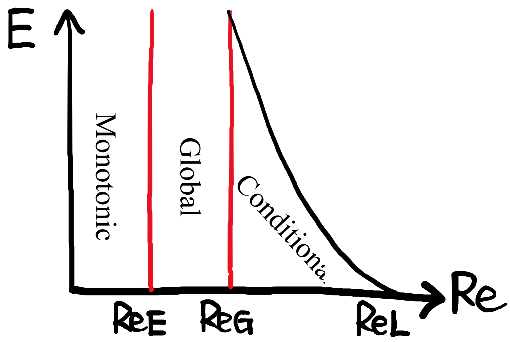


8. Critical Reynolds numbers

There are several critical Re defined based on different types of stability

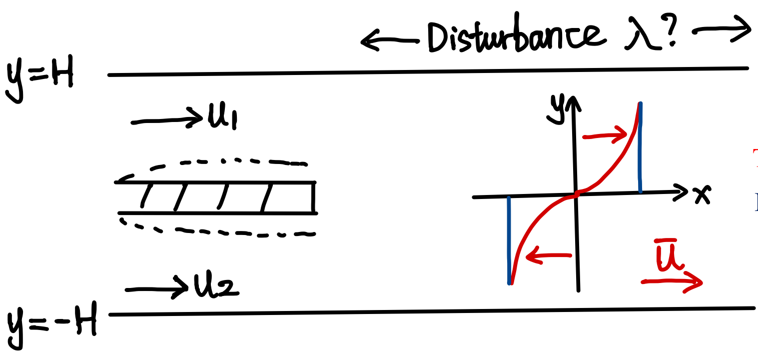
- For $\mathrm{Re} < \mathrm{Re}_\mathrm{E}$, the flow is monotonically stable
- For $\mathrm{Re} < \mathrm{Re}_\mathrm{G}$, the flow is globally stable
- For $\mathrm{Re} < \mathrm{Re}_\mathrm{L}$, the flow is linearly stable (i.e., conditionally stable)
  (i.e., when $\mathrm{Re} > \mathrm{Re}_\mathrm{L}$ there exists at least one infinitesimal disturbance that is unstable)
- For $\mathrm{Re} < \mathrm{Re}_\mathrm{T}$, the flow will relaminarize, only if Re_G does not correspond to the lowest Re for sustained turbulence

	Re_E	Re_G	Re_T	Re_L
Pipe	81.5	/	2000	∞
Channel	49.6	/	1000	5772
Plane Couette	20.7	125	360	∞



9. Global and local stability

Consider a wall shear layer. There is no well-posed global (all space) stability problem for the Navier-Stokes equation (NSE). Typically, we make a local approximation, move the local axis with $\bar{U} = (U_1 + U_2)/2$, and study the temporal growth of a spatially developing problem.



True profile relative to $\bar{U}(y)$

Inviscid variation

10. Open and closed domains

Even if $H \to \infty$, the flow is still open as $x \to \infty$. We make the closed and local approximation using a quasi-periodic model. In 3D, we seek a solution of the form

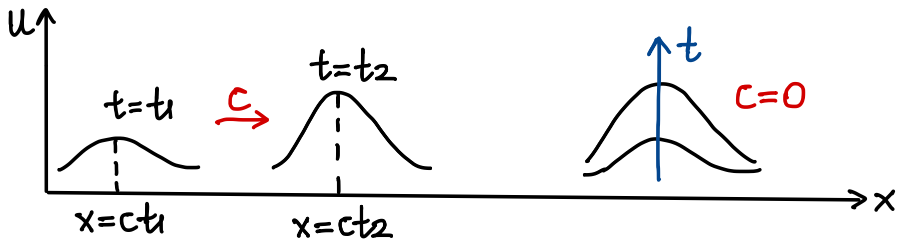
$$\boldsymbol{u}(x, y, z, t) = \boldsymbol{u}(x + n\lambda_x, y, z + m\lambda_z, t), \qquad n, m \in \mathbb{N}^+$$

The problem is then solved within the domain

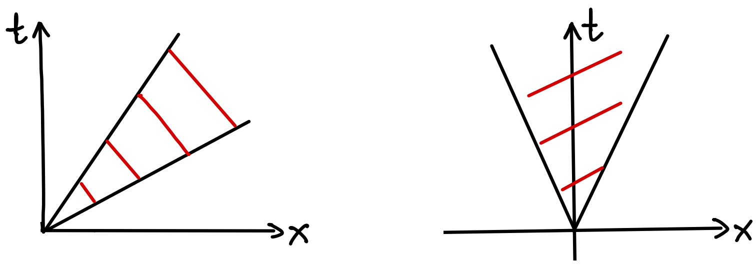
$$x \in \left(-\frac{\lambda_x}{2}, \frac{\lambda_x}{2}\right), \qquad y \in (-H, H), \qquad z \in \left(-\frac{\lambda_z}{2}, \frac{\lambda_z}{2}\right), \qquad t \geq 0$$

11. Convective & absolute instability

Assume $\boldsymbol{u}(\boldsymbol{x}, 0)$ has a compact support and $|\boldsymbol{u}| \sim e^{st}$ with Re $s > 0$ for some disturbances or modes. The solution $\boldsymbol{u}(\boldsymbol{x}, t) \to 0$ as $t \to 0$ for fixed $\boldsymbol{x}$, but $\boldsymbol{u}(\boldsymbol{x}, t) \to \infty$ as $t \to \infty$ if we move along $\boldsymbol{x} = \boldsymbol{c}t$ with group velocity $\boldsymbol{c} \neq 0$. This is a convective instability which dies away at any fixed point, but grow if we follow it with the group velocity $\boldsymbol{c}$. If there exists a mode with group velocity $\boldsymbol{c} = \boldsymbol{0}$ and thus $\boldsymbol{u}(\boldsymbol{x}, t) \to \infty$ as $t \to \infty$ for fixed $\boldsymbol{x}$, then it is absolutely unstable.



It is believed that local absolute instability implies global instability, the latter mathematically hard to describe. The implications of convective instability are not so clear. The wedge of instability can be used to describe convective instability. The wedge orientation corresponds to group velocity $\boldsymbol{c}$, while the width depends on operator's properties.



For an example of a simple 1D flow governed by the linear equation

$$\frac{\partial v}{\partial t} + \mathcal{L}\left(\frac{\partial}{\partial x}\right) v = 0$$

$\mathcal{L}$ is a linear differential (polynomial) operator. We seek ansatz of the form e^{ikx+st} to study the temporal evolution. The non-trivial solution satisfies

$$s + \mathcal{L}(ik) = 0, \qquad s = s(k)$$

Take an initial condition (I.C.) as

$$v(x,0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V(\xi)\, e^{ik(x-\xi)}\,\mathrm{d}\xi\,\mathrm{d}k$$

Then for $t > 0$ the solution is simply

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V(\xi)\, e^{ik(x-\xi)} e^{s(k)t}\,\mathrm{d}\xi\,\mathrm{d}k$$

For an arbitrary x, the absolute or convective instability depends on the limit of v as $t \to \infty$.

$$\lim_{t\to\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V(\xi)\, e^{-ik(x-\xi)} e^{s(k)t}\,\mathrm{d}\xi\,\mathrm{d}k$$

Suppose a δ-function perturbation source $V(\xi) = \delta(\xi)$, at $x = 0$ we have

$$v(0,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{s(k)t}\,\mathrm{d}k\,, \qquad v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{ikx+s(k)t}\,\mathrm{d}k$$

Given $\mathcal{L}(f) = -(1+f^2)$ as an example, we have

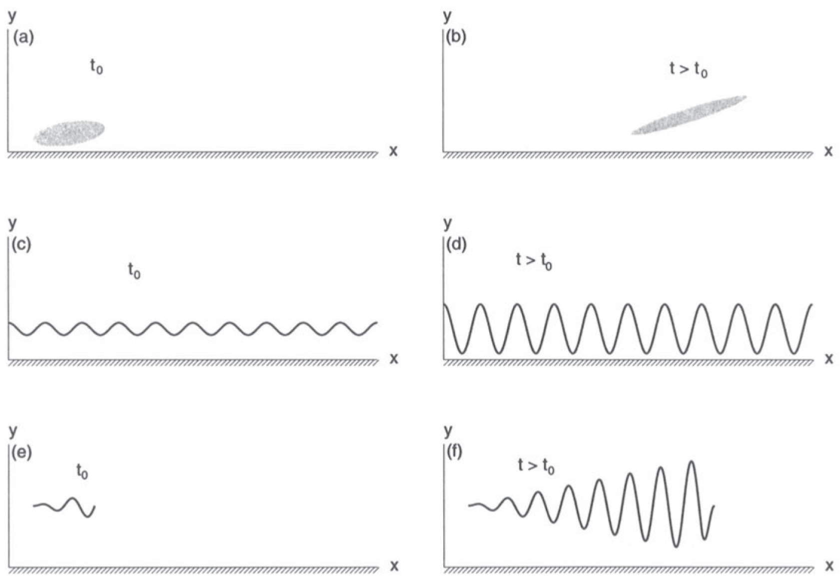
$$s = 1-k^2, \qquad v(x,t) \sim \int_{-\infty}^{\infty} e^{(1-k^2)t+ikx}\,\mathrm{d}k \sim \frac{1}{\sqrt{t}}\exp\left(t-\frac{x^2}{4t}\right)$$

This corresponds to an absolute instability.

12. Spatial and temporal evolution

The stability problem can be formulated in space and/or time for disturbance growth

- Temporal evolution of a spatially localized disturbance
- Temporal evolution of a globally periodic disturbance
- Spatial evolution of a temporally oscillatory source

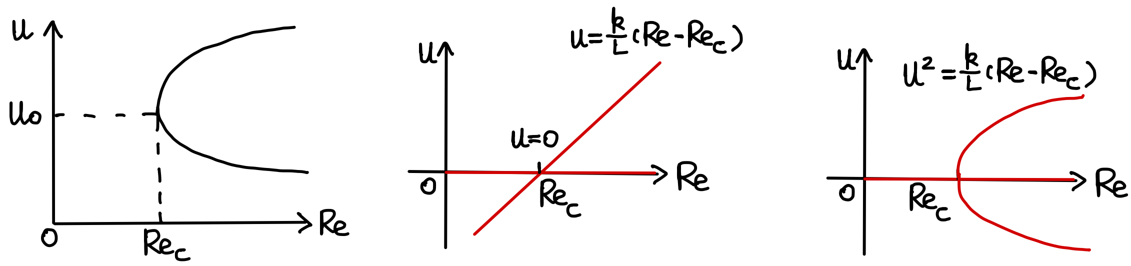


Comments

- ♦ A flow with no disturbance is not practical in laboratory.
- ♦ A flow may have one or more unstable disturbances, then it is an unstable flow. However, the most unstable disturbance may not be practically achievable / observable.
- ♦ A stable base flow may evolve until it becomes unstable. After growth it may then evolve to another stable flow (e.g., saturation of nonlinearity) or it may become turbulent.

➢ Bifurcation

Bifurcation indicates the change in the number or qualitative character of the solutions as some parameters change. Local bifurcations are considered through local stability of equilibrium.



1. Turning point / Saddle bifurcation

Consider that for some components of velocity $\boldsymbol{u}$ at some point as a function of Re, with some constants $k > 0$, $L \neq 0$, u_0 and critical Reynolds number $\mathrm{Re_c}$, we have

$$a = k(\mathrm{Re} - \mathrm{Re_c}), \qquad a - L(u - u_0)^2 = 0$$

At the turning point, the velocity component begins to show different paths

$$u = u_0 \pm \sqrt{\frac{k}{L}(\mathrm{Re} - \mathrm{Re_c})}$$

Below $\mathrm{Re_c}$ there is no solution, at $\mathrm{Re_c}$ there is one solution, and above $\mathrm{Re_c}$ two solutions.

2. Transcritical bifurcation

Consider the following form

$$au - Lu^2 = 0 \qquad \Longrightarrow \qquad u = 0, \qquad u = \frac{a}{L} = \frac{k(\mathrm{Re} - \mathrm{Re_c})}{L}$$

There are always two solutions except at the transcritical point $\mathrm{Re} = \mathrm{Re_c}$.

3. Pitchfork bifurcation

This is typical for flows with symmetry in u with the form

$$au - Lu^3 = 0 \qquad \Longrightarrow \qquad u = 0, \qquad u = \pm\sqrt{\frac{k}{L}(\mathrm{Re} - \mathrm{Re_c})}$$

➢ Normal mode analysis

Assume that a solution can be found to the temporal stability problem. The decomposition of the fields is

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x},t)=\boldsymbol{U}(\boldsymbol{x})+\boldsymbol{u}(\boldsymbol{x},t),\qquad \tilde{p}(\boldsymbol{x},t)=P(\boldsymbol{x})+p(\boldsymbol{x},t)$$

Consider the solution of the form

$$\boldsymbol{u}(\boldsymbol{x},t)=\widehat{\boldsymbol{u}}(\boldsymbol{x})e^{st},\qquad p(\boldsymbol{x},t)=\hat{p}(\boldsymbol{x})e^{st},\qquad s=\sigma+i\omega$$

Then the linearized NS equation becomes

$$s\widehat{\boldsymbol{u}}+\boldsymbol{U}\cdot\nabla\widehat{\boldsymbol{u}}+\widehat{\boldsymbol{u}}\cdot\nabla\boldsymbol{U}=-\nabla\hat{p}+\frac{1}{\mathrm{Re}}\nabla^2\widehat{\boldsymbol{u}},\qquad \nabla\cdot\widehat{\boldsymbol{u}}=0,\qquad \widehat{\boldsymbol{u}}|_{\partial V}=\boldsymbol{0}$$

The trivial solution $\widehat{\boldsymbol{u}}=\boldsymbol{0}$ and $\hat{p}=0$ always works, while there may exist non-zero solutions for special values of s. If the equation has non-trivial isolated and proper solutions (i.e., no singularities), then these solutions are the eigenvectors with eigenvalues s. Generally, each solution (normal mode) may be treated separately as it satisfies the linear system.

Write modes in terms of physical variables that enter the problem (e.g., Re, wavenumber k)

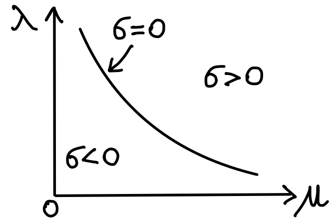
$$s=s(k_i,\lambda,\mu)$$

The set s is the discrete spectrum with mode label $k_1,k_2,\cdots,k_N$. The mode number N can be either finite ($N<\infty$) or infinite ($N\to\infty$). Then $\widehat{\boldsymbol{u}}(\boldsymbol{x}_i;k_i,\lambda,\mu)$ is the set of eigenvectors.

- If $\sigma>0$ for a given mode, then the **flow** is linearly unstable as a general disturbance can contain this mode.
- If $\sigma=0$ for a given mode, then the mode is neutrally stable.
- If $\sigma<0$ for a given mode, then the mode is asymptotically stable as the corresponding disturbance will decay.
- If $\sigma<0$ for **all** modes, then the **flow** is asymptotically stable.
- A mode is marginally stable if $\sigma(k_i;\lambda,\mu)=0$ for some values of (λ,μ) but then $\sigma>0$ for some neighboring values.

Marginal stability

Consider the neutral curve defined by $\sigma(k_i;\lambda,\mu)=0$. It is also a marginal curve because $\sigma>0$ nearby. In general, there may be no, one or many branches of the neutral curve for a given mode.



The imaginary part of σ gives the frequency ω of the mode

- If $\omega(k_i;\lambda,\mu)\neq 0$ at marginal stability, then the mode is overstable (oscillatory mode).
- If $\omega(k_i;\lambda,\mu)=0$ at marginal stability, there is an exchange of instability, indicating a base flow different from $\boldsymbol{U}(\boldsymbol{x})$, corresponding to a bifurcation of the steady flow solution.

$$\boldsymbol{U}(\boldsymbol{x}) \to \boldsymbol{U}(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})$$

Examples include convection cells when flow is heated from below, steep 2D water waves.

Generally, the eigenmodes are complex. A real disturbance can be obtained from taking the real part, or adding with its complex conjugate

$$\mathrm{Re}\{\hat{\boldsymbol{u}}(\boldsymbol{x})e^{st}\}, \qquad \hat{\boldsymbol{u}}(\boldsymbol{x})e^{st} + \hat{\boldsymbol{u}}^*(\boldsymbol{x})e^{s^*t}$$

Initial value problem

Consider the Laplace transform

$$\mathcal{L}\{\boldsymbol{u}(\boldsymbol{x},t)\} = \bar{\boldsymbol{u}}(\boldsymbol{x},s) = \int_0^{+\infty} \boldsymbol{u}(\boldsymbol{x},t)e^{-st}\,\mathrm{d}t\,, \qquad \mathcal{L}\left\{\frac{\partial \boldsymbol{u}}{\partial t}\right\} = s\bar{\boldsymbol{u}}(\boldsymbol{x},s) - \boldsymbol{u}(\boldsymbol{x},0)$$

For the linearized NS equation

$$s\bar{u}_i(\boldsymbol{x},s) + \cdots - \frac{1}{\mathrm{Re}}\nabla^2\bar{u}_i(\boldsymbol{x},s) = u_i(\boldsymbol{x},0), \qquad \frac{\partial \bar{u}_i}{\partial x_i} = 0, \qquad \bar{u}_i|_{\partial V} = 0$$

This can be solved using the Green's function

$$\bar{u}_i(\boldsymbol{x},s) = \int_V \bar{G}_{ij}(\boldsymbol{x},s;\boldsymbol{x}')\,u_j(\boldsymbol{x}',0)\,\mathrm{d}\boldsymbol{x}'$$

The Green's function satisfies

$$s\bar{G}_{ij}(\boldsymbol{x},s;\boldsymbol{x}') + \cdots - \frac{1}{\mathrm{Re}}\nabla^2\bar{G}_{ij}(\boldsymbol{x},s;\boldsymbol{x}') = \delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')$$

$\bar{G}_{ij}$ is the i-th component of $\bar{u}_i(\boldsymbol{x},s)$ when the source is $u_j(\boldsymbol{x},0) = \delta(\boldsymbol{x}-\boldsymbol{x}')$, or the impulse response of the linearized equation. The inverse Laplace transform is

$$u_i(\boldsymbol{x},t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{u}_i(\boldsymbol{x},s)\,e^{st}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\int_V \bar{G}_{ij}(\boldsymbol{x},s;\boldsymbol{x}')\,u_j(\boldsymbol{x}',0)\,\mathrm{d}\boldsymbol{x}'\right]e^{st}\,\mathrm{d}s$$
$$= \int_V G_{ij}(\boldsymbol{x},t;\boldsymbol{x}')\,u_j(\boldsymbol{x}',0)\,\mathrm{d}\boldsymbol{x}'$$

The time-domain Green's function is

$$G_{ij}(\boldsymbol{x},t;\boldsymbol{x}') = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{G}_{ij}(\boldsymbol{x},s;\boldsymbol{x}')\,e^{st}\,\mathrm{d}s$$

The properties of the eigen spectrum of $\boldsymbol{u}$ depends on the mathematical properties of $\bar{\boldsymbol{G}}(s)$.

- If all the singularities of $\bar{\boldsymbol{G}}(s)$ are poles, then there is a **discrete, complete spectrum**. This is typically associated with **self-adjoint operators**. The initial value problem is equivalent to the asymptotic stability one. Denote A_{ij} as the residue at pole k, we have

$$G_{ij}(\boldsymbol{x},t;\boldsymbol{x}') = \sum_k A_{ij}(\boldsymbol{x},\boldsymbol{x}';k)\,e^{s_k t}$$

- If there is a branch point / cut or essential singularity, then there is a **continuous spectrum**. Denote Γ as the branch cut given by $s = s(k)$, we have

$$G_{ij}(\boldsymbol{x}, t; \boldsymbol{x}') = \int_{\Gamma} \bar{G}_{ij}(\boldsymbol{x}, s(k); \boldsymbol{x}')\, e^{s(k)t}\, \mathrm{d}k$$

  The continuous spectrum often leads to algebraic rather than exponential growth in time.

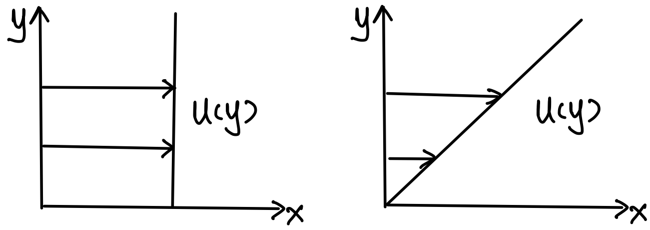
In general, the solution to the initial value problem is

$$\begin{aligned} u_i(\boldsymbol{x}, t) &= \int_V \left[\sum_k A_{ij}(\boldsymbol{x}, \boldsymbol{x}'; k)\, e^{s_k t} + \int_{\Gamma} \bar{G}_{ij}(\boldsymbol{x}, s(k); \boldsymbol{x}')\, e^{s(k)t}\, \mathrm{d}k \right] u_j(\boldsymbol{x}', 0)\, \mathrm{d}\boldsymbol{x}' \\ &= \sum_k \phi_k(\boldsymbol{x})\, e^{s_k t} + \int_{\Gamma} \phi(\boldsymbol{x}, k)\, e^{s(k)t}\, \mathrm{d}k \end{aligned}$$

This is the sum of proper normal modes and continuous spectrum. We call a mode proper and normal when s_k is independent of $\boldsymbol{x}$ and A_{ij} is separable as $A_{ij} = \phi(\boldsymbol{x})\, \psi_{ij}(\boldsymbol{x}')$.

Inviscid Linear Stability Analysis

➢ Parallel flows



Consider a base flow $U_i = U(y)\,\delta_{i1}$. The linearized momentum and continuity equations are

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} = -\frac{\partial p}{\partial x}, \qquad \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Velocity / Vorticity formulation

Take the divergence of momentum equation and use the continuity, and we can obtain the Poisson's equation for pressure (prime for y-derivative)

$$\nabla^2 p = -2U'\frac{\partial v}{\partial x}$$

The y-momentum equation then becomes

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x}\right] v = 0$$

Note that to obtain this result, the commutator of operators $U\partial_x$ and ∇^2 is not trivial. We can also obtain the vorticity equation in y-direction

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta = -U'\frac{\partial v}{\partial z}, \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

The boundary conditions can be stated as $v = 0$ at a solid wall and/or the far field. The initial conditions are

$$\eta(x, y, z, 0) = \eta_0(x, y, z), \qquad v(x, y, z, 0) = v_0(x, y, z)$$

The velocity / vorticity formulation gives a complete description of the evolution in both time and space of an arbitrary infinitesimal disturbance in an inviscid fluid.

The advantage of this formulation using normal velocity and vorticity is the absence of pressure term. If pressure is needed, it can be recovered by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p = \frac{\partial^2 v}{\partial t \partial y} + U\frac{\partial^2 v}{\partial x \partial y} - U'\frac{\partial v}{\partial x}$$

Modal solutions

Consider a disturbance with streamwise and spanwise wavenumbers α and β

$$\boldsymbol{u}(\boldsymbol{x},t) = \widehat{\boldsymbol{u}}(y)\, e^{i(\alpha x+\beta z-\alpha c t)}, \qquad c = c_{\mathrm{r}} + i c_{\mathrm{i}}$$

To obtain bounded solutions at $x, z \to \infty$, we have real α and β. This represents a propagating wave in the xz-plane with direction (α, β). The phase speed and the growth/decay rate are

$$c_p = \frac{\alpha c_{\mathrm{r}}}{\sqrt{\alpha^2+\beta^2}}, \qquad \boldsymbol{u} \propto \widehat{\boldsymbol{u}}(y)\, e^{\alpha c_i t}$$

Based on the growth rate, the stability of the flow can be analyzed as

- Flow is linearly stable if $\alpha c_i \le 0$ for all real wavenumbers α and β.
- Flow is unstable if $\alpha c_i > 0$ for at least one pair of real wavenumbers α and β.

The governing equations then become an ODE system (eigenvalue problem)

$$i\alpha(U-c)\hat{u} + U'\hat{v} = -i\alpha\hat{p}, \qquad i\alpha(U-c)\hat{v} = -\hat{p}'$$
$$i\alpha(U-c)\widehat{w} = -i\beta\hat{p}, \qquad i(\alpha\hat{u}+\beta\widehat{w}) + \hat{v}' = 0$$

For rigid boundaries, we have $\hat{v} = 0$ at a solid wall and/or the far field. The solution leads to the eigenvalue relation of the form $\mathcal{F}(\alpha,\beta,c) = 0$. In addition to this discrete spectrum, the singularity caused by $U - c = 0$ gives the continuous spectrum.

Squire's transformation & Equivalent 2D problem

We can rotate the xz-coordinate to align in the wavenumber direction (α,β)

$$\tilde{\alpha} = \sqrt{\alpha^2+\beta^2}, \qquad \tilde{\alpha}\tilde{u} = \alpha\hat{u} + \beta\widehat{w}, \qquad \tilde{p} = \frac{\tilde{\alpha}}{\alpha}\hat{p}, \qquad \tilde{v} = \hat{v}$$

This transformation reduces the problem to the equivalent 2D problem, which is the same by directly setting $\beta = 0$ and $\widehat{w} = 0$.

$$i\tilde{\alpha}(U-c)\tilde{u} + U'\tilde{v} = -i\tilde{\alpha}\tilde{p}, \qquad i\tilde{\alpha}(U-c)\tilde{v} = -\tilde{p}', \qquad i\tilde{\alpha}\tilde{u} + \tilde{v}' = 0$$

Squire's theorem

For rigid boundaries (as the transform does not properly handle BCs otherwise), each unstable 3D disturbance corresponds to a more unstable 2D case.

To prove this, denote $c = f(\alpha)$ as the solution to the 2D problem with $\beta = 0$ and $\widehat{w} = 0$. Then we have $c = f(\tilde{\alpha})$ as the solution to the equivalent 2D problem. Therefore, the original 3D problem has the solution

$$c = f\left(\sqrt{\alpha^2+\beta^2}\right)$$

When $\beta \neq 0$, we have $\tilde{\alpha} > \alpha$. For each unstable 3D mode with a growth rate $\alpha c_i > 0$, the corresponding 2D mode has $\tilde{\alpha} c_i > \alpha c_i > 0$, which is more unstable.

Stream function formulation for 2D problem

From Squire's theorem, we only need to consider 2D disturbances, which can be described by the stream function

$$\psi'(x,y,t) = \phi(y)\, e^{i\alpha(x-ct)}, \qquad \hat{u} = \frac{\mathrm{d}\phi}{\mathrm{d}y} = \phi', \qquad \hat{v} = -i\alpha\phi$$

The continuity is automatically satisfied. The momentum equations become

$$-(U-c)\,\phi' + U'\phi = \hat{p}, \qquad (U-c)(\phi'' - \alpha^2\phi) - \phi U'' = 0$$

They are invariant under the transformation

$$u \to -u, \qquad v \to v, \qquad x \to -x, \qquad t \to -t, \qquad p \to -p$$

So, we have the first pair of solutions

$$\phi(y)\, e^{i\alpha(x-ct)} \to -\phi(y)\, e^{-i\alpha(x-ct)}$$

As the coefficients of the Rayleigh equation are real, if ϕ is the eigenfunction with stable eigenvalue c satisfying $\alpha c_i > 0$, then there is another unstable mode ϕ^* that corresponds to eigenvalue c^*. We can thus take $|c_i| > 0$ as the condition for instability. Note that neutral modes with $c_i = 0$ are also useful because they may adjoin unstable solutions.

- Classical inviscid stability theorems

We focus on the temporal stability of parallel shear flows. Assume that the base flow is parallel with $U_i = U(y)\,\delta_{1i}$, the fluid is inviscid (Newtonian, incompressible, etc.) and perturbations are small so that we can neglect nonlinear terms.

Rayleigh equation

Start with the wall-normal velocity equation

$$\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U''\frac{\partial}{\partial x}\right] v = 0$$

In modal form $v(\boldsymbol{x},t) = \hat{v}(y)\, e^{i(\alpha x + \beta y - \alpha c t)}$, we have the Rayleigh equation

$$(U-c)(D^2 - k^2)\hat{v} - U''\hat{v} = 0, \qquad k^2 = \alpha^2 + \beta^2, \qquad D = \frac{\mathrm{d}}{\mathrm{d}y}$$

Boundary conditions are $\hat{v} = 0$ at $y = \pm 1$ for bounded flows or $\hat{v} = 0$ at $y = 0$ and $y \to \infty$ for semi-infinite domains. This is an eigenvalue problem for a second-order differential operator. The wave speed c is the complex eigenvalue and appears in pairs with c^*. As the boundary conditions are unchanged with the sign of α, so we take $\alpha \geq 0$ and the criterion for instability becomes $c_i > 0$ for some positive α.

Adjoint Rayleigh equation

In the operator form, Rayleigh equation becomes

$$\mathcal{L}\phi = [(U-c)(D^2 - \alpha^2) - U'']\,\phi = 0$$

Define an inner product

$$\langle \phi_1, \phi_2 \rangle = \int_{y_1}^{y_2} \phi_1^* \phi_2 \, \mathrm{d}y\,, \qquad \langle \phi_1, \phi_2 \rangle^* = \langle \phi_2, \phi_1 \rangle$$

Under the same boundary conditions, $\phi^\dagger$ and $\mathcal{L}^\dagger$ are called the adjoints of ϕ and $\mathcal{L}$ if we have

$$\langle \phi^\dagger, \mathcal{L}\phi \rangle = \langle \mathcal{L}^\dagger \phi^\dagger, \phi \rangle$$

Rayleigh equation is not self-adjoint. Its adjoint equation is

$$\mathcal{L}^\dagger \phi^\dagger = [(D^2 - \alpha^2)(U - c) - U''] \, \phi^\dagger = 0$$

Compare with the Rayleigh equation, we can obtain the self-adjoint form

$$\phi^\dagger = \frac{A\phi}{U - c}, \qquad D\left[(U-c)^2 D\left(\frac{\phi}{U-c}\right)\right] - \alpha^2 (U - c)\phi = 0$$

The adjoint eigenvalues are conjugates of the eigenvalues. The eigenfunctions may not be orthogonal to each other (e.g., for a non-normal operator), but the adjoint eigenfunctions are orthogonal to each other, form a complete basis, and orthogonal to all the eigenfunctions except the one that shares a conjugate eigenvalue (bi-orthogonality).

Rayleigh's inflection point criterion

Rayleigh criterion gives a necessary condition for instability. It does not require specification of a full velocity profile. The criterion states that if there exist an unstable mode with $c_i > 0$, then we must have $U''(y) = 0$ somewhere in the domain.

Proof. Multiply the Rayleigh equation by v^* and integrate in y from $[-1,1]$. Assume $c_i > 0$ so that the Rayleigh equation is not singular. For this unstable mode, we have

$$\int_{-1}^{1} \left(\hat{v}^* D^2 \hat{v} - \alpha^2 |\hat{v}|^2 - \frac{U''}{U - c} |\hat{v}|^2 \right) \mathrm{d}y = 0$$

The first term is manipulated using integration by part with the wall boundary conditions

$$\int_{-1}^{1} \hat{v}^* D^2 \hat{v} \, \mathrm{d}y = -\int_{-1}^{1} D\hat{v}^* D\hat{v} \, \mathrm{d}y = -\int_{-1}^{1} |D\hat{v}|^2 \, \mathrm{d}y$$

We now have

$$\int_{-1}^{1} (|D\hat{v}|^2 + \alpha^2 |\hat{v}|^2) \, \mathrm{d}y + \int_{-1}^{1} \frac{U''}{U - c} |\hat{v}|^2 = 0$$

The first term is real and positive definite, while the second term is generally complex as

$$\int_{-1}^{1} \frac{U''}{U - c} |\hat{v}|^2 = \int_{-1}^{1} U'' |\hat{v}|^2 \frac{U - c_r + i c_i}{|U - c|^2} \, \mathrm{d}y$$

The only way for the imaginary part to be zero is that U'' changes sign in the domain.

Alternatively, any profile without an inflection point is linearly, inviscidly stable as Re $\to \infty$.

Fjørtoft's criterion (1950)

Conditions for a linearly unstable base profile can be further constrained by investigating the real part of the same equation. For a monotonic $U(y)$, a necessary condition for instability is

$$U''(U - U_s) < 0, \qquad U_s = U(y_s)$$

where y_s is the inflection point with $U''(y_s) = 0$. This implies that the inflection point has to be a maximum of the spanwise mean vorticity.

Proof. The real part of the equation is

$$\int_{-1}^{1} \frac{U''(U - c_r)}{|U - c|^2} |\hat{v}|^2 \, \mathrm{d}y = -\int_{-1}^{1} (|D\hat{v}|^2 + \alpha^2 |\hat{v}|^2) \, \mathrm{d}y$$

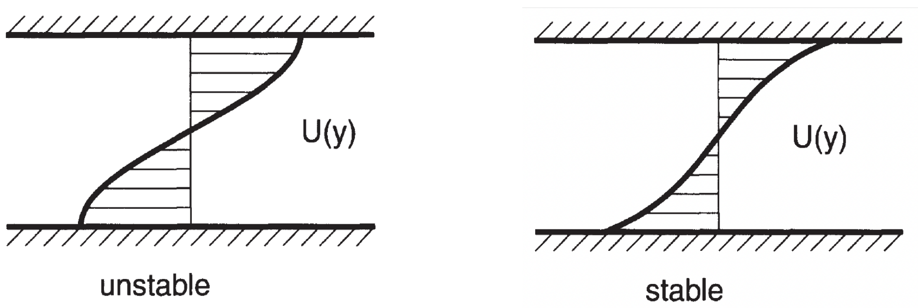
As we know from the imaginary part

$$\int_{-1}^{1} \frac{U''}{|U - c|^2} |\hat{v}|^2 \, \mathrm{d}y = 0 \qquad \Longrightarrow \qquad (c_r - U_s) \int_{-1}^{1} \frac{U''}{|U - c|^2} |\hat{v}|^2 \, \mathrm{d}y = 0$$

we can add it to the LHS and obtain

$$\int_{-1}^{1} \frac{U''(U - U_s)}{|U - c|^2} |\hat{v}|^2 \, \mathrm{d}y = -\int_{-1}^{1} (|D\hat{v}|^2 + \alpha^2 |\hat{v}|^2) \, \mathrm{d}y$$

The RHS is negative, so we must have $U''(U - U_s) < 0$ somewhere in the domain.



In terms of spanwise vorticity, the condition can be written as

$$\frac{\partial \omega_z}{\partial y}(U - U_s) > 0, \qquad \omega_z = -\frac{\partial U}{\partial y}, \qquad U'' = -\frac{\partial \omega_z}{\partial y}$$

Howard's semicircle theorem

The eigenvalues of the Rayleigh equation are confined to a disk of radius R, center $c_r + ic_i$

$$R = \frac{U_{\max} - U_{\min}}{2}, \qquad c_r = \frac{U_{\max} + U_{\min}}{2}, \qquad c_i = 0$$

Proof. Recall the self-adjoint form of the adjoint Rayleigh equation

$$\hat{v}^\dagger = \frac{\hat{v}}{U-c}, \qquad D[(U-c)^2 D\hat{v}^\dagger] - \alpha^2 (U-c)^2 \hat{v}^\dagger = 0$$

Multiply the conjugate of $\hat{v}^\dagger$ and integrate over the domain

$$\int_{-1}^{1} (U-c)^2 Q \,\mathrm{d}y = 0, \qquad Q = \left|D\hat{v}^\dagger\right|^2 + \alpha^2 \left|\hat{v}^\dagger\right|^2$$

The real and imaginary parts become

$$\int_{-1}^{1} [(U-c_r)^2 - c_i^2]\, Q \,\mathrm{d}y = 0, \qquad 2c_i \int_{-1}^{1} (U-c_r)\, Q \,\mathrm{d}y = 0$$

The imaginary part implies that $U - c_r$ must change sign in the domain. For a general $c_i \neq 0$, we have $U_{\min} < c_r < U_{\max}$. Now we obtain

$$\int_{-1}^{1} UQ \,\mathrm{d}y = c_r \int_{-1}^{1} Q \,\mathrm{d}y$$

Using this expression, the real part gives

$$\int_{-1}^{1} (U^2 - 2Uc_r + c_r^2 - c_i^2)\, Q \,\mathrm{d}y = 0, \qquad \int_{-1}^{1} U^2 Q \,\mathrm{d}y = (c_r^2 + c_i^2) \int_{-1}^{1} Q \,\mathrm{d}y$$

Since $U_{\min} \le U(y) \le U_{\max}$, using the above results, we have

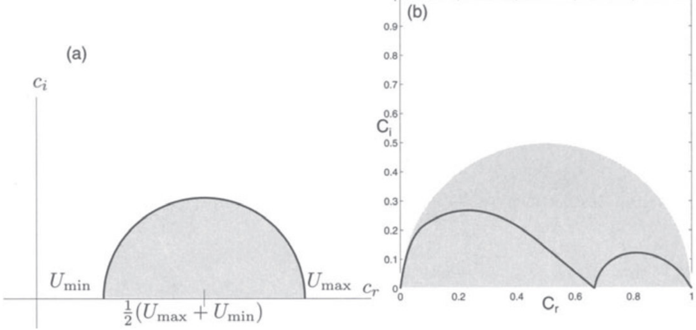
$$0 \ge \int_{-1}^{1} (U - U_{\min})(U - U_{\max})\, Q \,\mathrm{d}y$$

$$\ge \int_{-1}^{1} [(c_r^2 + c_i^2) - (U_{\min} + U_{\max})c_r + U_{\min}U_{\max}]\, Q \,\mathrm{d}y$$

This finally leads to

$$\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 \le \left(\frac{U_{\max} - U_{\min}}{2}\right)^2$$

As an example, consider the Bickley jet with a steady 2D laminar profile $U(y) = \operatorname{sech} y$. Even though the fluid is viscous, at high Re we can apply the inviscid approximation. The unstable eigenvalues (as α varies) fall within Howard's semicircle.



Conditions for sufficiency

Tollmien's heuristic argument: Rayleigh and Fjørtoft criteria are only sufficient for symmetric profiles in a channel, and monotonic profiles of boundary layer type. A sufficient condition is

$$\left.\frac{\partial c_i}{\partial(\alpha^2)}\right|_{\alpha_s} < 0$$

This implies that we have unstable $c_i > 0$ for α slightly lower than the neutral mode α_s.

Tollmien's counter example

Consider a sinusoidal base flow $U(y) = \sin y$ for $y \in [-h, h]$. We have $U''(y) = -U$, and the profile is monotonic for $h \leq \pi/2$. The inflection point is $y_s = n\pi$, and consider $h = \pi/2$, we have one inflection point at $y_s = 0$. The necessary Rayleigh and Fjørtoft conditions are both satisfied, but the flow is stable.

The issue lies in the marginally stable eigen solution, which exists in the general case

$$c = U_s = 0, \qquad \alpha = \alpha_s = \left[1 - \frac{n^2\pi^2}{(y_2 - y_1)^2}\right]^{\frac{1}{2}}$$

$$\phi = \phi_s = \sin\left[\frac{n\pi(y_2 - y_1)}{y_2 - y_1}\right], \qquad \text{for } |n| < \frac{y_2 - y_1}{\pi}, \quad n \in \mathbb{Z}$$

Hence, if $y_2 - y_1 < \pi$, the flow is stable as there only exists the trivial mode $\phi_s = 0$.

➢ Critical layers

A critical layer is defined as a location in the flow where $U(y_c) = c_r$, the wave speed is real and equal to the local mean velocity. We require $U'(y_c) \neq 0$, and note that $c \neq c_s$ necessarily. From the Rayleigh equation

$$(D^2 - k^2)\hat{v} - \frac{U''}{U - c}\hat{v} = 0$$

we notice that $U(y) = c$ is a regular singular point of logarithmic type in the complex plane, with exponents 0 and 1. For instability, $c_i \neq 0$ and there is no singularity, then it can be solved numerically. The local solution can be expressed in terms of a Frobenius series expanding the velocity profile around the critical layer

Tollmien's inviscid solutions

$$\hat{v}_1(y) = (y - y_c)P_1(y), \qquad \hat{v}_2(y) = P_2(y) + \frac{U_c''}{U_c'}\hat{v}_1(y)\ln(y - y_c)$$

Tollmien gave two linearly independent solutions, and $P_1(y)$ and $P_2(y)$ are analytic functions determined by boundary conditions. Usually, we choose $P_1(y_c) = P_2(y_c) = 1$.

- $\hat{v}_1(y)$ is the regular inviscid solution with $P_1(y_c) \neq 0$.
- $\hat{v}_2(y)$ is the singular inviscid solution with a logarithmic branch point at $y = y_c$, and we call $P_2(y)$ as the regular part of the singular solution.

For a real c, we also have a real y_c and we specify the value as

$$\ln(y - y_c) = \begin{cases} \ln|y - y_c|, & y > y_c \\ \ln|y - y_c| - i\pi, & y < y_c \end{cases}$$

The singular inviscid solution is discontinuous at the critical layer.

Streamlines near the critical layer

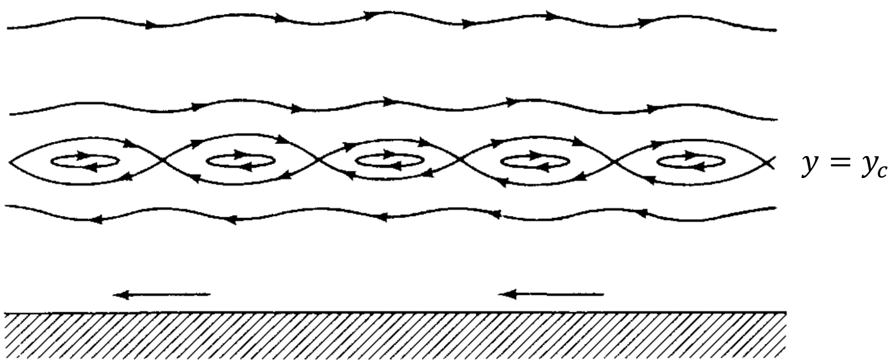
Assume a wave amplitude A moving with velocity c such that the motion is steady. Streamlines, streak lines and particle paths all coincide. Define a stream function as

$$\psi(y) = \int_{y_c}^{y} [U(y) - c]\, \mathrm{d}y + A\, \mathrm{Re}\{\phi e^{i\alpha\tilde{x}}\}, \qquad \tilde{x} = x - ct$$

Near the critical layer, we have

$$U - c \approx U_c'(y - y_c), \qquad \phi \approx \phi(y_c), \qquad \psi(y) \approx \frac{1}{2} U_c'(y - y_c)^2 + A\phi(y_c)\cos(\alpha\tilde{x})$$

The resulting field is the Kelvin's cat's eye pattern of the streamlines near the critical layer. There are two types of stagnation point, hyperbolic and center type. There is a concentration of vorticity at the cat's eye, but no instability.



Reynolds stress

The behavior of the Reynolds stress $\overline{uv}$ is helpful in understanding the structure of the critical layers. Consider the stress averaged in the x-direction

$$\tau(y) = -\overline{uv} = -\frac{\alpha}{2\pi} \int_0^{\frac{2\pi}{\alpha}} uv\, \mathrm{d}x$$

With the fluctuations in terms of the stream function, we have

$$u = \mathrm{Re}\{\phi' \, e^{i\alpha(x-ct)}\}, \qquad v = \mathrm{Re}\{-i\alpha\phi \, e^{i\alpha(x-ct)}\}$$

The Reynolds stress and its gradient are obtained as

$$\tau(y) = \frac{i\alpha}{4}(\phi\phi'^* - \phi^*\phi') \, e^{2\alpha c_i t}, \qquad \frac{\mathrm{d}\tau}{\mathrm{d}y} = \frac{1}{2}\alpha c_i \frac{U''|\phi|^2}{|U-c|^2} e^{2\alpha c_i t}$$

Note that the derivation uses the Rayleigh equation to substitute ϕ''

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0$$

Suppose that an unstable mode exists for some α and that as $\alpha \to \alpha_s$ (neutral mode), we have $c_i \to 0, c_r \to U(y_c) = U_c$ and $U \to c$ as $y \to y_c$. Then $\mathrm{d}\tau/\mathrm{d}y = 0$ everywhere except possibly at $y = y_c$ where we need to evaluate

$$\frac{\mathrm{d}\tau}{\mathrm{d}y} = \lim_{\substack{c_i \to 0 \\ c_r \to U_c}} \frac{1}{2}\alpha c_i \frac{U''|\phi|^2}{(U-c_r)^2 + c_i^2}$$

Near the critical layer $y = y_c$, we can write

$$\frac{\mathrm{d}\tau}{\mathrm{d}y} = \lim_{c_i \to 0} \frac{1}{2}\alpha c_i \frac{U''|\phi|^2}{U_c'^2(y-y_c)^2 + c_i^2}$$

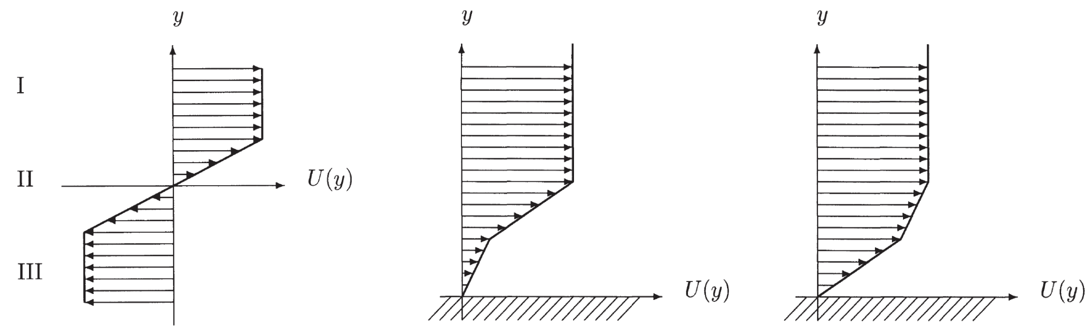
Integrate across the critical layer and let $c_i \to 0$ from above, the discontinuity in τ is

$$\Delta\tau = \tau(y_c^+) - \tau(y_c^-) = \lim_{c_i \to 0} \frac{1}{2}\alpha U_c''|\phi|^2 \int_{y_c-\varepsilon}^{y_c+\varepsilon} \frac{c_i \, \mathrm{d}y}{U_c'^2(y-y_c)^2 + c_i^2} = \frac{1}{2}\alpha\pi \frac{U_c''}{U_c'}|\phi|^2$$

As we have $\tau(y_2) = \tau(y_1) = 0$, if U is monotonic, then there can only be one jump in τ (i.e. one critical layer). Because $\mathrm{d}\tau/\mathrm{d}y = 0$ for the neutral mode, we have $\Delta\tau = 0$ and $U_c'' = 0$, which corresponds to an inflection point at the critical layer. In general, we have $\sum \Delta\tau = 0$ but for each critical layer $\Delta\tau \neq 0$.

- Piecewise linear profiles

Parallel flow stability problems can be numerically solved in full now. The historical approach of approximating the base flow using piecewise linear profiles is still instructive because it leads to analytical forms for the dispersion relationship $c(\alpha, \beta)$ and eigenfunctions. Piecewise linear profiles mean that they can have discontinuities either in U or U'.



Within each linear part, we have $U'' = 0$ and the Rayleigh equation becomes

$$(U-c)(D^2-k^2)\hat{v} = 0$$

However, we need two jump conditions at the discontinuity. Integrate the Rayleigh equation across the discontinuity $y = y_D$, we have

$$D[(U-c)D\hat{v} - U'\hat{v}] = (U-c)k^2\hat{v}, \qquad [(U-c)D\hat{v} - U'\hat{v}]_-^+ = 0$$

This states that the pressure is continuous across the discontinuity. The first jump condition for eigenfunction $\hat{v}$ can be written as

$$\hat{p} = \frac{i\alpha}{k^2}[U'\hat{v} - (U-c)D\hat{v}], \qquad [\![(U-c)D\hat{v} - U'\hat{v}]\!] = 0$$

The second jump condition is obtained as

$$-\frac{k^2\hat{p}}{i\alpha(U-c)^2} = D\left[\frac{\hat{v}}{U-c}\right], \qquad \left[\!\!\left[\frac{\hat{v}}{U-c}\right]\!\!\right] = 0 \quad \text{or} \quad \left[\!\!\left[\frac{\phi}{U-c}\right]\!\!\right] = 0$$

For continuous U, this is equivalent to matching vertical velocity (i.e. the interface is material).

Example: Mixing layer

The base flow is given as $U(y) = \pm 1$ above and below the middle layer described as

$$U(y) = y, \qquad -1 \le y \le 1$$

The boundary and jump conditions at discontinuities $y = \pm 1$ are stated as

$$\hat{v}|_{y\to\pm\infty} \to 0, \qquad [\![(U-c)D\hat{v} - U'\hat{v}]\!] = 0, \qquad \left[\!\!\left[\frac{\hat{v}}{U-c}\right]\!\!\right] = 0$$

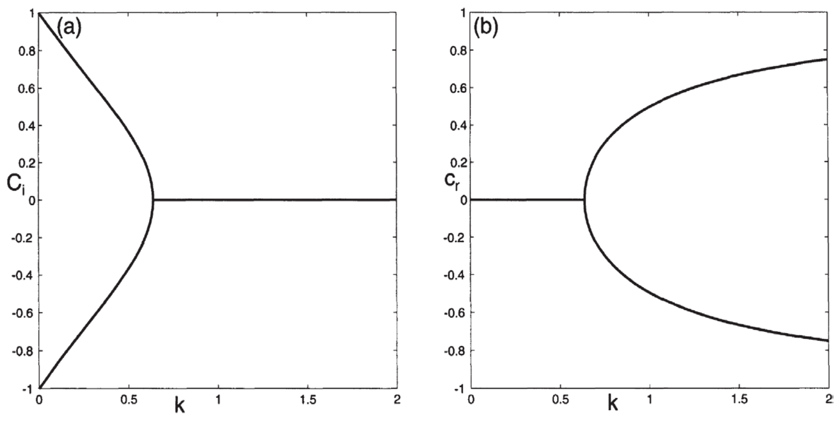
The solution in each region has the form

$$\hat{v}_I = Ae^{-ky}, \qquad \hat{v}_{II} = Be^{-ky} + Ce^{ky}, \qquad \hat{v}_{III} = De^{ky}$$

By matching the jump conditions at $y = \pm 1$, we have

$$c = \pm\sqrt{\left(1 - \frac{1}{2k}\right)^2 - \frac{e^{-4k}}{4k^2}}$$

For larger $k > 0.6392$, the eigenvalues are real and all disturbances are neutral. Conversely, for smaller $0 \le k < 0.6392$, the eigenvalues are purely imaginary and the flow is unstable.

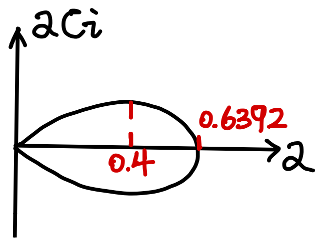


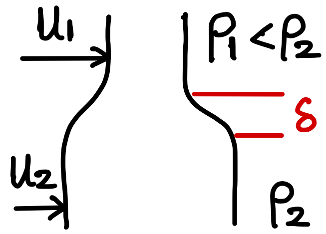
Kelvin-Helmholtz instability

The limit of $k \to 0$ is equivalent to the limit of having a zero thickness of shear (vortex sheet). This corresponds to the Kelvin-Helmholtz instability.

$$\lim_{k\to 0} c^2(k) = \lim_{k\to 0} \frac{4k^2 - 4k + 1 - e^{-4k}}{4k^2} = -1, \qquad c = \pm i$$

This implies that wavelike perturbations on an unbounded vortex sheet in an inviscid fluid do not show any dispersion. All wave components share the same zero speed. The maximal growth rate occurs at $k = 0.4$ (or denoted as α). $\alpha = 0.6392$ corresponds to $\alpha\delta = 1.28$, with the shear layer thickness $\delta = 2$ here. This prediction shows good agreement with the real flow without approximation, where the most amplified wavelength is $\lambda \approx 7.85\ \delta$.





In general, the Kelvin-Helmholtz family of instabilities treat parallel flows with (continuous) profiles $u(y)$ and $\rho(y)$ over depth δ. Now we consider a discontinuity at $y = 0$ with two layers wand the perturbed interface as $y = \eta(x,t)$. For an irrotational flow, we have

$$\nabla^2\phi_2 = 0, \qquad \nabla^2\phi_1 = 0$$

As $z \to \pm\infty$, the velocity field becomes just the base flow

$$\nabla\phi_2 \to U_2\hat{\boldsymbol{\imath}}, \qquad \nabla\phi_1 \to U_1\hat{\boldsymbol{\imath}}$$

At the interface, the material assumption / requirement gives the condition on vertical velocity

$$\frac{\partial\phi}{\partial y} = \frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}, \qquad \text{at } \ y = \eta(x,t)$$

The pressure is also continuous across the interface. The Bernoulli equation gives

$$\rho_1\left[c_1 - \frac{1}{2}|\nabla\phi_1|^2 - \frac{\partial\phi_1}{\partial t} - g\eta\right] = \rho_2\left[c_2 - \frac{1}{2}|\nabla\phi_2|^2 - \frac{\partial\phi_2}{\partial t} - g\eta\right]$$

For the base flow, we similarly have

$$\rho_1\left(c_1 - \frac{1}{2}U_1^2\right) = \rho_2\left(c_2 - \frac{1}{2}U_2^2\right)$$

This states the non-linear problem for stability of the base flow.

To linearize the problem, in the two domains we consider the perturbation to the base flow

$$\phi_2 = U_2 x + \phi_2', \qquad \phi_1 = U_1 x + \phi_1'$$

Then the kinematic and dynamic conditions for the perturbation become

$$\frac{\partial\phi_i'}{\partial y} = \frac{\partial\eta}{\partial t} + U_i\frac{\partial\eta}{\partial x}, \qquad \rho_1\left[U_1\frac{\partial\phi_1'}{\partial x} + \frac{\partial\phi_1'}{\partial t} + g\eta\right] = \rho_2\left[U_2\frac{\partial\phi_2'}{\partial x} + \frac{\partial\phi_2'}{\partial t} + g\eta\right]$$

For small η, we can evaluate these conditions at $y = 0$. Normal mode representation gives the eigenvalue problem using the solution form $\phi_i' = \hat{\phi}_i \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}+st}$ and we have

$$\rho_1[kg + (s + ik_x U_1)^2] = \rho_2[kg - (s + ik_x U_2)^2], \qquad k^2 = k_x^2 + k_y^2$$

Neutral stability can be found if

$$kg(\rho_1^2 - \rho_2^2) \geq k_x^2 \rho_1 \rho_2 (U_1 - U_2)^2$$

There are several special cases:

- Deep water surface gravity waves: $\rho_2 = 0, U_1 = U_2 = 0$, neutral stability.
- Internal gravity waves: $\rho_1 > \rho_2, U_1 = U_2 = 0$, heavier fluids below, neutral stability.
- Rayleigh-Taylor instability: $\rho_1 < \rho_2, U_1 = U_2 = 0$, acceleration-driven instability. This is deserved in inertial confinement facilities (ICFs) where accelerating shocks / interfaces become unstable. The growth rate in the linear regime is given as

$$s = \pm\sqrt{g'k}, \qquad g' = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} g$$

- Shear-driven instability: $\rho_1 = \rho_2$

- Dispersive effects and wave packets

The eigenvalue relationship is given by $c(\alpha, \beta)$ with streamwise $\alpha = k_x$ and spanwise $\beta = k_z$. The wave speed of an actual disturbance is a complex function of wavenumbers. Consider the dispersion of a perturbation consisting of discrete modes as

$$v(x, y, z, t) = \frac{1}{4\pi} \iint \hat{v}(y, \alpha, \beta) \, e^{i(\alpha x + \beta z - \omega t)} \, \mathrm{d}\alpha \mathrm{d}\beta$$

Assume $\beta = 0$ and the asymptotic behavior is analyzed from the stationary phase

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(\alpha x - \omega t) = 0, \qquad x = \frac{\mathrm{d}\omega}{\mathrm{d}\alpha} t = c_g^x t$$

- Initial value problem revisited

Define the wall-normal vorticity as $\eta(\boldsymbol{x}, t)$ for 2D parallel flow. The modal representation is

$$v(\boldsymbol{x}, t) = \hat{v}(y, t) \, e^{i(\alpha x + \beta z)}, \qquad \eta(\boldsymbol{x}, t) = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \hat{\eta}(y, t) \, e^{i(\alpha x + \beta z)}$$

The velocity and vorticity equations are

$$\left[\left(\frac{\partial}{\partial t} + i\alpha U\right)(D^2 - k^2) - i\alpha U''\right] \hat{v} = 0, \qquad \left(\frac{\partial}{\partial t} + i\alpha U\right) \hat{\eta} = -i\beta U' \hat{v}$$

The boundary conditions are $\hat{v} = 0$ at solid walls and in far field. We also have

$$\hat{u} = \frac{i}{k^2}(\alpha D\hat{v} - \beta\hat{\eta}), \qquad \hat{w} = \frac{i}{k^2}(\beta D\hat{v} + \alpha\hat{\eta})$$

Lift-up effect

When $\beta \neq 0$ (without Squire transformation), integrating the equation for η gives

$$\hat{\eta} = \hat{\eta}_0 e^{-i\alpha U t} - i\beta U' e^{-i\alpha U t} \int_0^t \hat{v}(y, t')\, e^{i\alpha U t'}\, \mathrm{d}t'$$

The first term is the advection of vorticity by the base flow. The second term is the lift-up of fluid by the normal velocity in the presence of shear (when $\beta \neq 0$ and $U' \neq 0$). Consider a fluid parcel transported by v in $+y$-direction for $U' > 0$, and being replaced by a parcel at larger y with larger velocity. The parcels will appear as streaky fluctuations relative to the base flow. Even for decaying $\hat{v}$, this mechanism can lead to large disturbance amplitudes in the horizontal component.

Algebraic instability

When $\alpha = 0$ the growth can be calculated explicitly. In this case, the Rayleigh equation implies that $\hat{v}$ is not a function of time. From the previous solution, we have

$$\hat{\eta} = \hat{\eta}_0 - i\beta U' \hat{v}_0 t$$

This behavior is known as algebraic instability, as the growth is not exponential.

Viscous Instability

- Governing equations for parallel flows

The viscous linear stability equations are

$$\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2-U''\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^4\right]v=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}-\frac{1}{\mathrm{Re}}\nabla^2\right)\eta=-U'\frac{\partial v}{\partial z}$$

The pressure equation is still

$$\nabla^2 p=-2U'\frac{\partial v}{\partial x}$$

The boundary conditions are $v=v'=0$ and $\eta=0$ at the solid walls and in the far field.

Orr-Sommerfeld and Squire equations

With the normal mode representation

$$v(\boldsymbol{x},t)=\hat{v}(y)\,e^{i(\alpha x+\beta z-\omega t)},\qquad \eta(\boldsymbol{x},t)=\hat{\eta}(y)\,e^{i(\alpha x+\beta z-\omega t)}$$

we obtain the Orr-Sommerfeld (O-S) equation for $\hat{v}$ and the Squire equation for $\hat{\eta}$

$$\left[(-i\omega+i\alpha U)(D^2-k^2)-i\alpha U''-\frac{1}{\mathrm{Re}}(D^2-k^2)^2\right]\hat{v}=0$$

$$\left[(-i\omega+i\alpha U)-\frac{1}{\mathrm{Re}}(D^2-k^2)\right]\hat{\eta}=-i\beta U'\hat{v}$$

The boundary conditions are $\hat{v}=D\hat{v}=0$ and $\hat{\eta}=0$ at the solid walls and in the far field. These are viscous extensions of Rayleigh and inviscid Squire equations. The vorticity equation is coupled to velocity unless $\beta=0$ or $U'=0$. Note that O-S equation is fourth order.

Eigenmodes

There are two sets of eigenmodes for the OS-S system. The **Orr-Sommerfeld modes** are found by solving the OS equation, then using $\hat{v}$ mode to solve the inhomogeneous Squire equation, The eigenmodes and eigenvalues are described as

$$\{\hat{v}_n,\ \hat{\eta}_n^p,\ \omega_n\}_{n=1}^N$$

Here $\hat{\eta}^p$ is the particular solution of the Squire equation. In a bounded domain, the eigenvalues are discrete and infinite in number. In an unbounded domain, there is usually a finite number of discrete modes complemented by a continuous spectrum. The **Squire modes** have $\hat{v}=0$, meaning that OS modes are identically zero. Then solve the homogeneous Squire equation

$$\{\hat{v}=0,\ \hat{\eta}_m,\ \omega_m\}_{m=1}^M$$

In general, the set of ω_n is different from the set of ω_m. Solution methods include expansion in orthogonal basis functions, finite difference formulation and shooting methods.

OS-S system in vector form

In vector form, we have

$$-i\omega\begin{bmatrix} k^2-D^2 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{v} \\ \hat{\eta}\end{bmatrix}+\begin{bmatrix}\mathcal{L}_{OS} & 0 \\ i\beta U' & \mathcal{L}_{SQ}\end{bmatrix}\begin{bmatrix}\hat{v} \\ \hat{\eta}\end{bmatrix}=0$$

The OS and Squire operators are defined as

$$\mathcal{L}_{OS}=i\alpha U(k^2-D^2)+i\alpha U''+\frac{1}{\mathrm{Re}}(k^2-D^2)^2, \qquad \mathcal{L}_{SQ}=i\alpha U+\frac{1}{\mathrm{Re}}(k^2-D^2)$$

Denote the eigenvector as $\widehat{\boldsymbol{q}}$, and we thus obtain

$$\boldsymbol{L}\widehat{\boldsymbol{q}}=i\omega\boldsymbol{M}\widehat{\boldsymbol{q}}, \qquad \boldsymbol{L}=\begin{bmatrix}\mathcal{L}_{\mathrm{OS}} & 0 \\ i\beta U' & \mathcal{L}_{\mathrm{SQ}}\end{bmatrix}, \qquad \boldsymbol{M}=\begin{bmatrix}k^2-D^2 & 0 \\ 0 & 1\end{bmatrix}$$

Note that $\boldsymbol{M}$ is a positive definite operator.

- Squire's theorem

Squire's transformation for viscous flows

Compare 3D and 2D ($\beta=0$) OS equations

$$\left[(U-c)(D^2-k^2)-U''-\frac{1}{i\alpha\mathrm{Re}}(D^2-k^2)^2\right]\hat{v}=0$$

$$\left[(U-c)(D^2-\alpha_{2\mathrm{D}}^2)-U''-\frac{1}{i\alpha_{2\mathrm{D}}\mathrm{Re}_{2\mathrm{D}}}(D^2-\alpha_{2\mathrm{D}}^2)^2\right]\hat{v}=0$$

The solutions to these equations will be identical if

$$\alpha_{2\mathrm{D}}=k=\sqrt{\alpha^2+\beta^2}, \qquad \alpha_{2\mathrm{D}}\mathrm{Re}_{2\mathrm{D}}=\alpha\mathrm{Re} \qquad \Longrightarrow \qquad \mathrm{Re}_{2\mathrm{D}}=\frac{\alpha}{k}\mathrm{Re}<\mathrm{Re}$$

For an unstable 3D disturbance, there will be a 2D disturbance that is unstable at lower $\mathrm{Re}_{2\mathrm{D}}$.

Squire modes

It can be shown that the Squire modes are always damped, $c_i<0$ for all α,β,Re.

Squire's theorem

The onset of (linear) instability is given by

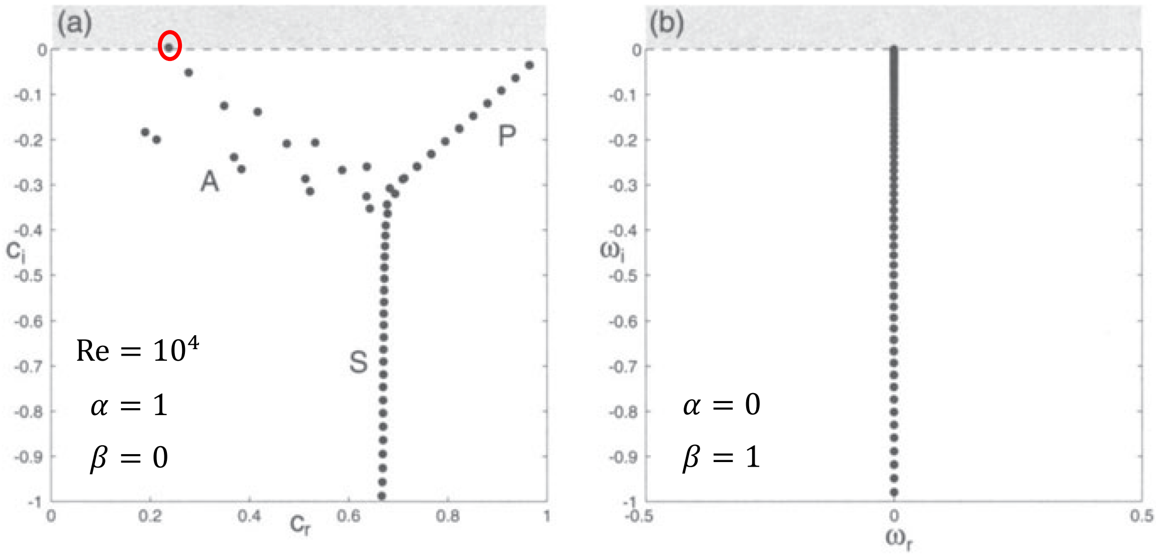
$$\mathrm{Re}_{\mathrm{c}}=\min_{\alpha,\beta}\mathrm{Re}(\alpha,\beta)=\min_{\alpha}\mathrm{Re}(\alpha,0)$$

- Eigenvalue spectra of OS equation

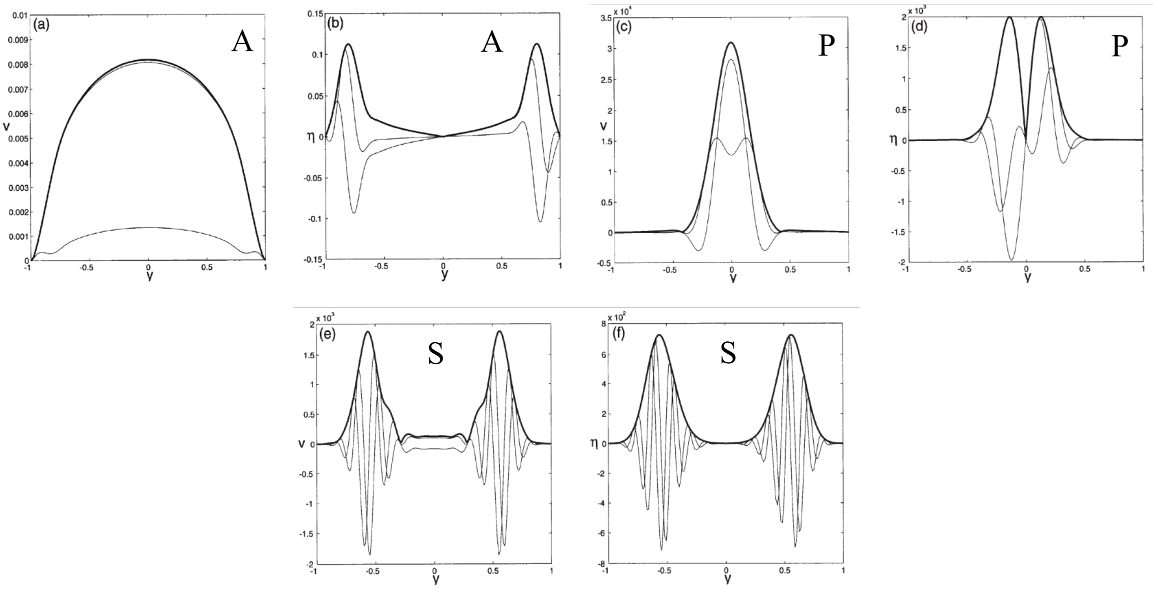
Poiseuille channel flow

At $\mathrm{Re}=10^4$, numerical results are shown for two cases. When $\alpha\neq 0$, A branch consists of wall modes with $c_r\to 0$, while the P branch consists of center modes with $c_r\to 1$ for centerline. The S branch consists of modes with $c_r\approx 2/3$ (bulk velocity) and are highly damped.

There is one slightly unstable eigenmode on the A branch, which is called Tollmien-Schlichting (TS) wave. This shows viscosity as a destabilizing factor even though the base flow does not satisfy Rayleigh's inflection point criterion.



The eigenfunctions of A mode, P mode and S mode are shown below for $\alpha = \beta = 1$. The real and imaginary parts (thin lines) indicate the phase variation.



For special case $\alpha = 0$, there is only one branch with $\omega_r = 0$ and always $\omega_i < 0$ (damped). Eigenvalues can be determined analytically. The Squire modes and O-S modes are

$$\omega_{\mathrm{sq}} = \frac{i}{\mathrm{Re}}\left[\beta^2 - \left(n - \frac{1}{2}\right)^2 \pi^2\right], \qquad \omega_{\mathrm{os}} = -\frac{i}{\mathrm{Re}}(\beta^2 - \mu^2)$$

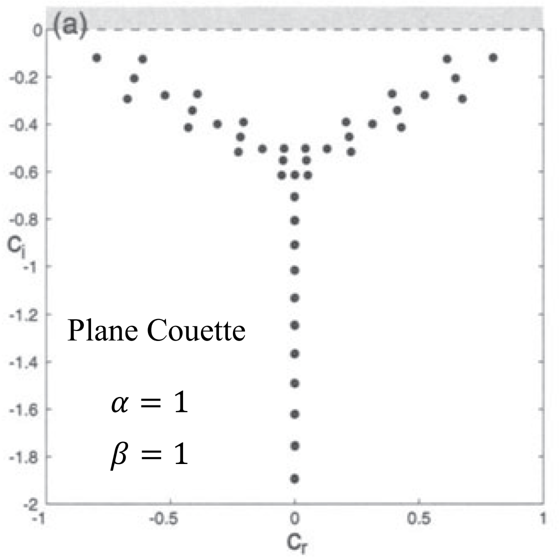
For O-S modes, μ is the solution of the transcendental equation $\mu \tan\beta + \beta \tan\mu = 0$ for odd modes and $\mu \cot\beta + \beta \cot\mu = 0$ for even modes.

Couette flow

Couette flow has a slightly different structure: No P branch but two A branches, which implies that each eigenfunction has a reflection about the centerline. Compare to Poiseuille flow where eigenfunctions come in symmetric and anti-symmetric pairs. Couette flow is linearly stable for any Re, with no eigenvalues having $c_i > 0$.

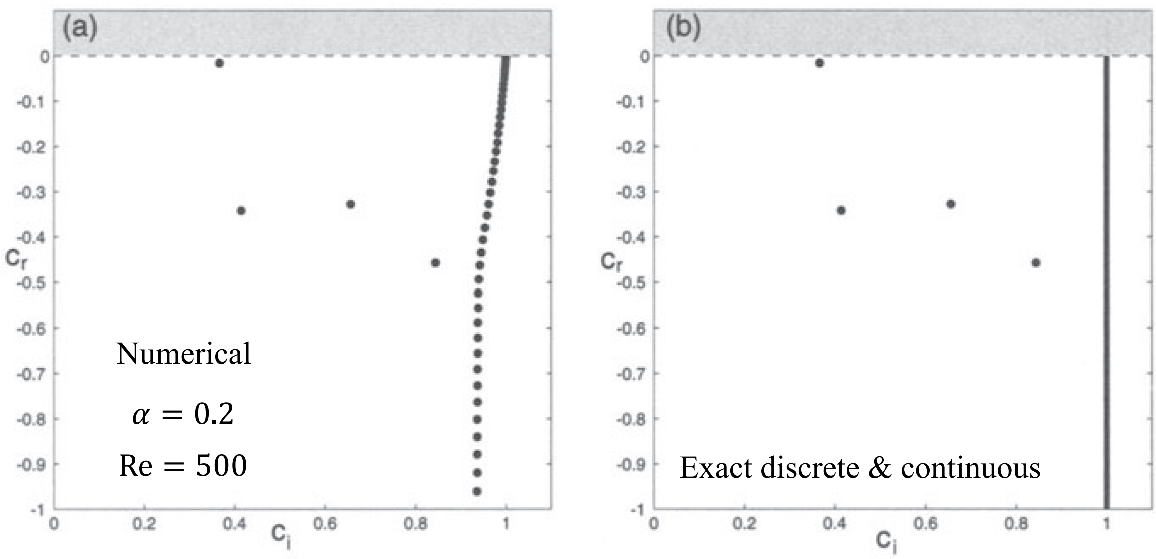
Hagen-Poiseuille (pipe) flow

Pipe has the same 3-branch structure as channel flow. The circular cross-section constrains that only integer azimuthal wavenumbers are allowed, which is denoted by n. As for Couette flow, pipe flow is also linearly stable.



Boundary layers (Blasius flow)

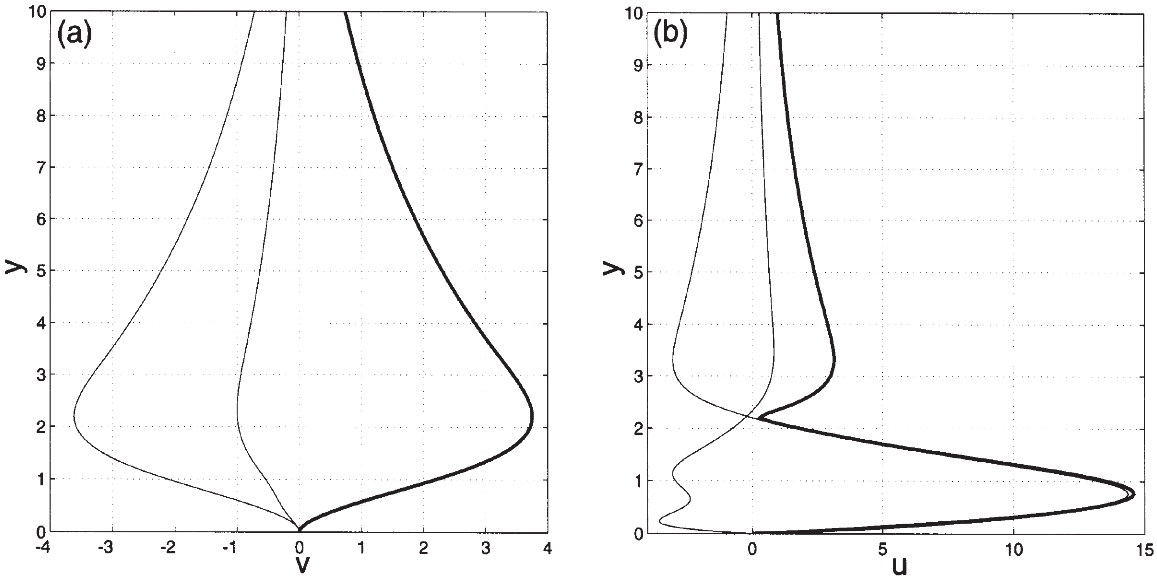
Boundary layers are not streamwise homogeneous in general, so the locally parallel flow assumption is required to use this same approach.



In essence, we assumes $U(x, y)$ and $V(x, y)$ with $V \ll U$ and $\partial U/\partial x \ll 1$ for the flow to be locally parallel. Consider the following Blasius solution

$$\frac{U(y)}{U_\infty} = f'(\eta), \qquad \eta = \frac{y}{\delta^*(x)}$$

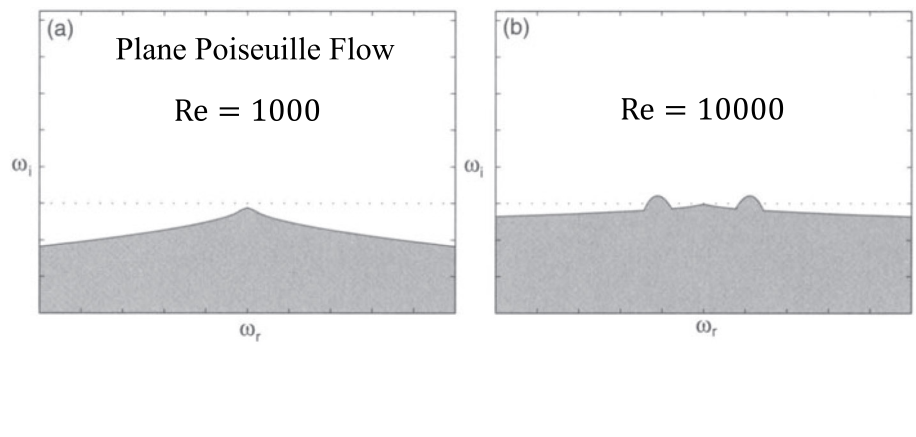
We solve for eigenvalues in this form, which consist of a few eigenvalues on A branch, and a continuous spectrum that represents a combination of P and S branches. The continuous part arises from the semi-infinite domain. The eigenfunction of the discrete mode shows that there is a phase change in u, which relates to the critical layer.



The Blasius boundary layer has at most one unstable mode (TS wave) on the A branch. Because of its low wave speed c_r, it is concentrated close to the wall.

➢ Spectrum of continuous stability operator

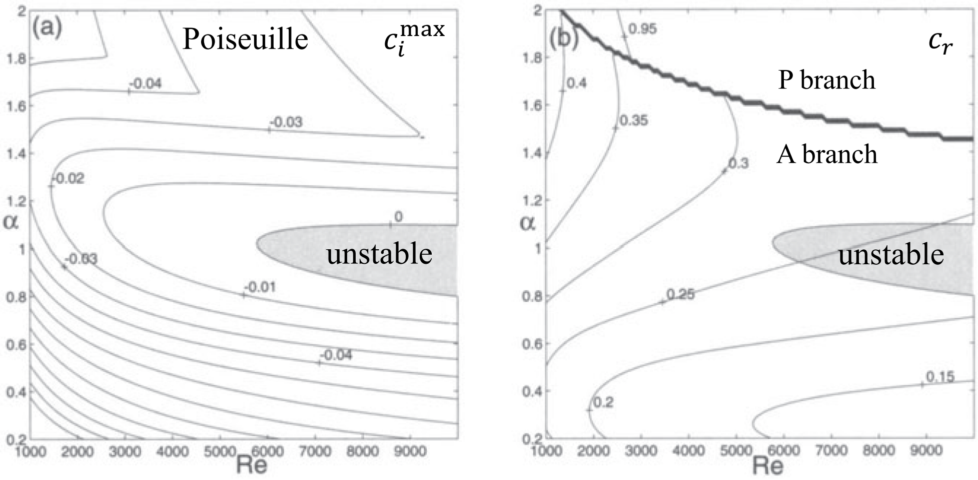
The above examples give rise to a discrete set of eigenvalues because the operator has been discretized at a specific pair of (α, β). Variation of (α, β) leads to a range of eigenvalues ω that constitutes the spectrum of the continuous linear stability operator. Taking the Fourier transform (discretizing in ω) leads to the discrete point spectrum. For boundary layer, the continuous spectrum is required, and the discretization by Fourier transform is not complete.



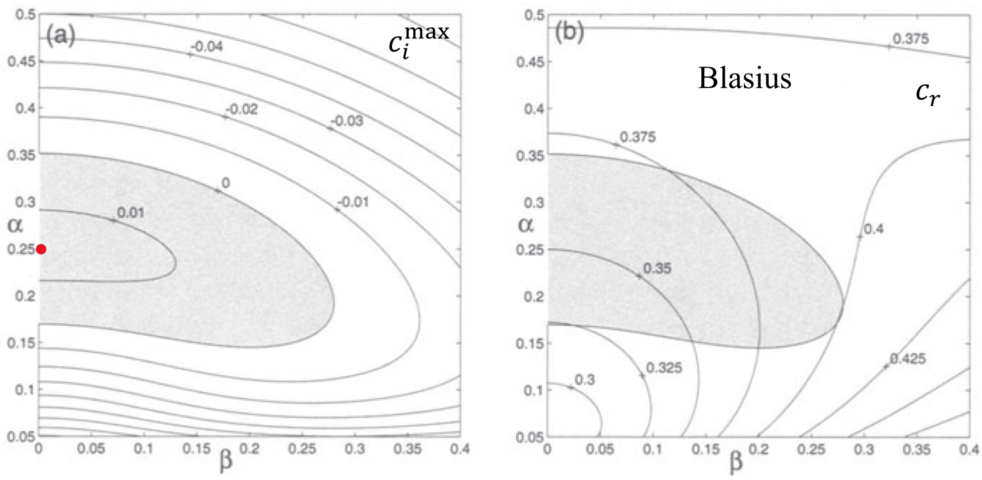
For pipe flows, the integer constraint on the azimuthal wavenumber n restricts the continuous operator spectrum to be a set of lines. Points in the complex plane that do not fall on one of the lines do not represent eigenvalues for any wave number combination.

- Neutral curves

When considering canonical flows, it is useful to find a representation of the regions of our parameters space that are unstable or stable, divided by the neutral curve. Squire's theorem implies that such a curve can be drawn in the (α, Re) space, focusing on 2D instability.



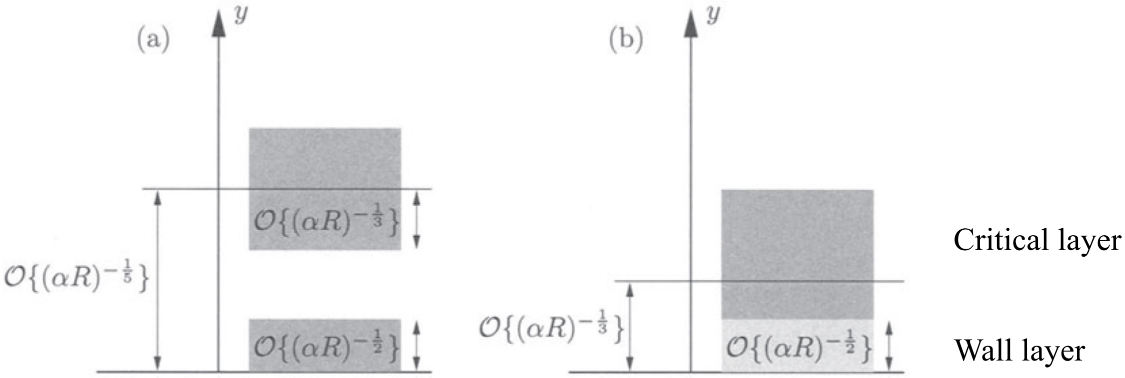
The neutral curve is the contour for $c_i = 0$. Within the neutral curve, unstable solutions exist. For Blasius boundary layer, $\mathrm{Re} = 519.4$, $\alpha_c = 0.303$, $c_{r,c} = 0.3965$. When $\mathrm{Re} > \mathrm{Re}_{\mathrm{crit}}$, the neutral curve has two branches (bifurcation). Variation of phase velocity c_r reflects A branch, except for the top right region where the most unstable mode is on the P branch.



We can also plot contours in the (α, β) space at a fixed Re. The maximum growth rate occurs on the line $\beta = 0$. Corrections for non-parallel flow can also be made.

➢ Viscous critical layers

For high Re, inviscid arguments hold for most of the domain, but the effects of viscosity locally remain important. Much of the discussion of inviscid critical layer holds over to the viscous case. Viscosity is important in resolving the logarithmic singularity at the critical layer and in meeting the wall no-slip boundary conditions.



The viscous modifications can occur in regions of the flow that are distinct or which overlap in y-direction. The scaling relations above are determined by dominant balance in each layer. Eventually, the critical layer singularity could also be resolved by a non-negligible non-linear term, or a combination with viscosity.

➢ Non-normality of the stability operator

A non-normal operator is one for which $\mathcal{L}\mathcal{L}^+ \neq \mathcal{L}^+\mathcal{L}$, i.e. it does not commute with its adjoint. One consequence is that the eigenfunctions of the operator will be non-orthogonal. **The OS operator ($\mathcal{L}_{OS}$) is non-normal**, leading to sensitivity of the eigen spectrum and the transient growth of perturbations (energy growth on a finite time horizon even in linearly stable flows).

Eigenvalue sensitivity

The discussion here does not involve numerical issues on determining the eigenvalue spectra. The eigenvalues of the OS-S system are sensitive to small perturbations of the equations, e.g., those due to discretization of continuous equations or finite-precision arithmetic. The sensitivity is a property of the linearized Navier-Stokes operator, namely its non-normality. The sensitivity is the size of perturbation to eigenvalue $\delta\lambda$ for a given perturbation to the operator.

$$\boldsymbol{\mathcal{L}q} = \boldsymbol{M}^{-1}\boldsymbol{Lq} = i\omega\boldsymbol{q} = \lambda\boldsymbol{q}$$

Denote $\boldsymbol{P}$ as the perturbation operator with $\|\boldsymbol{P}\| = \varepsilon \ll 1$. The modified equation is

$$(\boldsymbol{\mathcal{L}} + \boldsymbol{P})(\boldsymbol{q} + \delta\boldsymbol{q}) = (\lambda + \delta\lambda)(\boldsymbol{q} + \delta\boldsymbol{q})$$

The first-order terms in ε lead to

$$(\boldsymbol{\mathcal{L}} - \lambda\boldsymbol{I})\,\delta\boldsymbol{q} + \boldsymbol{Pq} = \delta\lambda\,\boldsymbol{q}$$

An upper bound on the eigenvalue change can be found from the energy norm

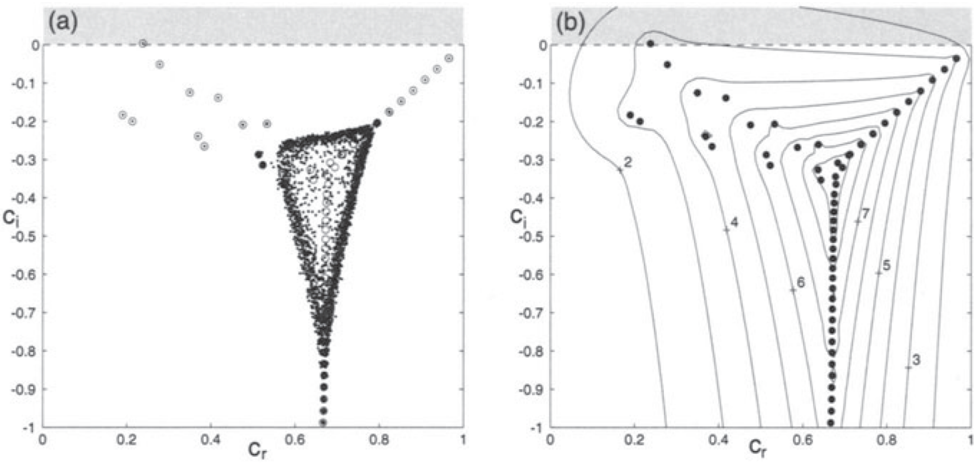
$$|\delta\lambda| \leq \|\boldsymbol{q}^{+}\|_E \|\boldsymbol{P}\|_E \|\boldsymbol{q}\|_E$$

Pseudo-spectrum

Trefethen (1992) introduced pseudo-spectra and pseudo-eigenvalues to characterize the spectra of perturbed, non-normal matrices and operators. Consider a randomly perturbed matrix $\tilde{A} = A + E$ with $\|E\| \leq \varepsilon$. The ε-pseudo eigenvalues z are the eigenvalues of the perturbed matrix $\tilde{A}$. They satisfy the following relationship

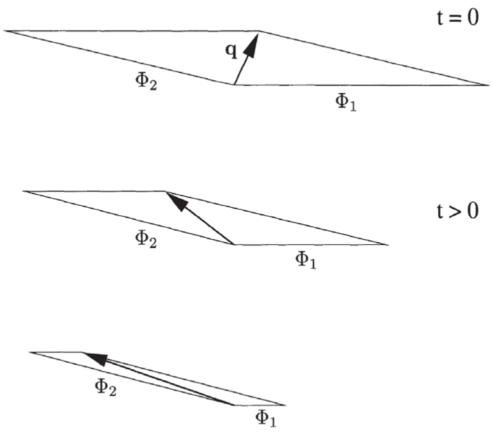
$$\|(z\boldsymbol{I} - \boldsymbol{A})^{-1}\| \geq \varepsilon^{-1}$$

The resolvent of $\boldsymbol{A}$ is defined as $R(z) = (z\boldsymbol{I} - \boldsymbol{A})^{-1}$ everywhere on the complex plane except at the eigenvalues of $\boldsymbol{A}$.



Transient growth in viscous initial value problem

There can be a transient growth of perturbation $\boldsymbol{q}$ in a finite time horizon from the superposition of two non-orthogonal vectors decaying at different rates as time evolves. This is the viscous equivalence to algebraic growth considered for inviscid parallel flows.



In the illustration, $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are eigenfunctions with a particular weighting that results in the perturbation vector $\boldsymbol{q}$. The magnitude of $\boldsymbol{q}$ grows with time at the beginning even though both eigenfunctions are decaying. Eventually all magnitudes decay to zero for a linearly stable operator. The angle between eigenvectors is equally important.

To study the magnitude of perturbation $\boldsymbol{q}$, recall the eigenvalue problem

$$\boldsymbol{\mathcal{L}q} = \boldsymbol{M}^{-1}\boldsymbol{Lq} = i\omega\boldsymbol{q}, \qquad \boldsymbol{q} = [\hat{v}, \hat{\eta}]^T$$

A formal solution of the initial value problem is

$$\frac{\partial \boldsymbol{q}}{\partial t} = i\boldsymbol{\mathcal{L}q}, \qquad \boldsymbol{q}(t) = e^{i\boldsymbol{\mathcal{L}}t}\boldsymbol{q}_o, \qquad e^{i\boldsymbol{\mathcal{L}}t} = \boldsymbol{I} + i\boldsymbol{\mathcal{L}}t - \frac{1}{2}\boldsymbol{\mathcal{L}}^2 t^2 + \cdots$$

The energy norm is applied as a measure of perturbation growth, which is defined as

$$E_v = \iint \mathcal{E}\, \mathrm{d}\alpha \mathrm{d}\beta = \iint \frac{1}{2k^2} \int_{-1}^{1} (|\mathrm{D}\hat{v}|^2 + k^2|\hat{v}|^2 + |\hat{\eta}|^2)\, \mathrm{d}y\, \mathrm{d}\alpha \mathrm{d}\beta$$

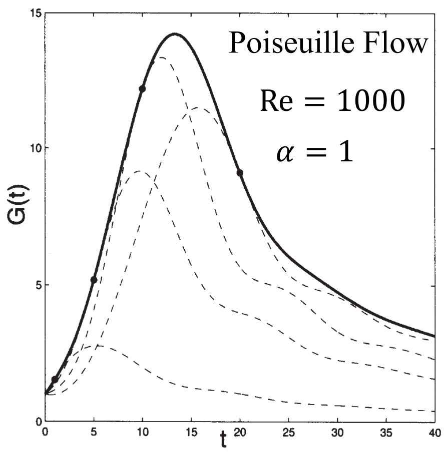
Note that $\mathcal{E}$ is the energy density in the Fourier space.

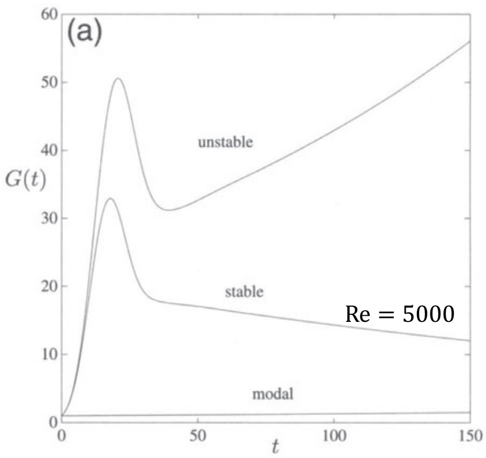
Optimal growth

Define the maximum possible amplification of the initial energy density as

$$G(t) = \max_{\boldsymbol{q}_o \neq \boldsymbol{0}} \frac{\|\boldsymbol{q}(t)\|^2}{\|\boldsymbol{q}_o\|^2} = \left\|e^{i\boldsymbol{\mathcal{L}}t}\right\|^2$$

The curve $G(t)$ is the envelope of the evolution of all possible initial conditions $\boldsymbol{q}_o$. $G_{\max}$ is the absolute maximum occurring at time $t = t_{\max}$. For $t \to \infty$, we have $G(t) \to 0$ for a linearly stable system. Optimal growth refers to the maximum possible growth.

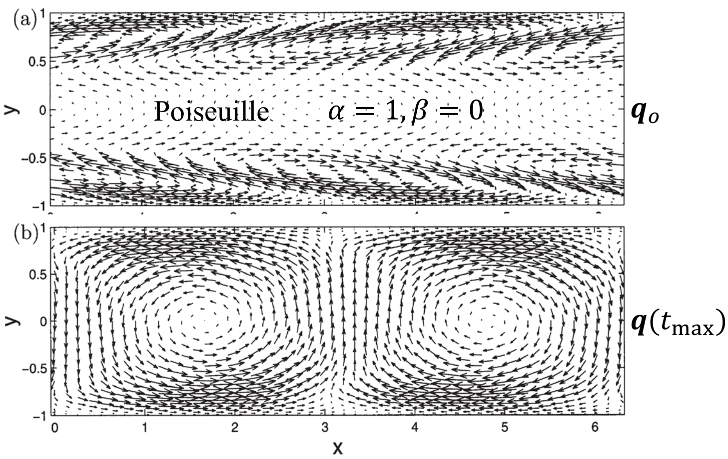


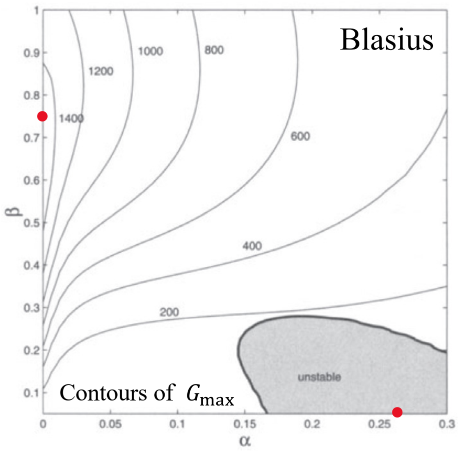


The maximum amplification occurs for an initial condition that is a combination of eigen-functions. Such an initial condition can lead to growth that is significantly higher than for a single eigenfunction for an unstable operator.

- The presence of a growing eigenfunction can directly lead to instability.
- If $\|\boldsymbol{q}_o\|$ is large, nonlinearity may kick in at time before $G_{\max}$.

Maximum amplification is obtained for $\alpha = 0$, but another mechanism is also important. The tilting of an initial condition that is upstream towards a vertical orientation at $t_{\max}$ corresponds to the transient growth. For $\alpha = 0$ perturbation, it simply strengthens to $t_{\max}$ and then decays.





Maximum transient amplification occurs at $\alpha = 0$, while maximum modal growth at $\beta = 0$.

Summary of optimal growth for different flows

	$G_{\max}$ (10^{-3})	$t_{\max}$	α	β
Plane Poiseuille	0.2 Re^2	0.076 Re	0	2.04
Plane Couette	1.18 Re^2	0.117 Re	35 / Re	1.6
Circular pipe	0.07 Re^2	0.048 Re	0	1
Blasius boundary layer	1.50 Re^2	0.778 Re	0	0.65

Further Examples of Stability and Instability

- Weakly nonlinear instability analysis

Consider a parallel flow with a perturbation of the form

$$\boldsymbol{u}'(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{U}(\boldsymbol{x}) = \sum_j \left[A_j(t)\,\boldsymbol{\phi}_j(\boldsymbol{x}) + A_j^*(t)\,\boldsymbol{\phi}_j^*(\boldsymbol{x})\right]$$

$\boldsymbol{\phi}_j$ are the eigenfunctions. For linear theory, we have

$$A_j(t) = a_j e^{s_j t}, \qquad s_j = \sigma_j + i\omega_j$$

There is no mechanism for interactions between different $A_j(t)$. For weakly nonlinear theory, we analyze some nonlinear interactions among $A_j(t)$. Assume that there exists a critical parameter R_c such that all disturbances are stable for $R < R_c$, and there is one normal mode with $\sigma_1 > 0$ when $R = R_c$. Usually, $\sigma_1 = k(R - R_c) + O[(R - R_c)^2]$, where the parameter R can be Reynolds number, Rayleigh number, etc.

Landau equation

Consider the weakly nonlinear equation for quantity A. There are many methods to derive this, but they rest on defining A as

$$A = \varepsilon \hat{A}(\xi,\tau) e^{st}, \qquad \varepsilon \ll 1, \qquad \tau = \varepsilon^2 t, \qquad \xi = \varepsilon\left(x - c_g t\right)$$

Note that τ denotes the long timescale and c_g is the group velocity of the neutral disturbances. Also $R - R_c = \varepsilon^2 \sigma_1$. Now A varies on two timescales: e^{st} and $\hat{A}(\xi,\tau)$. Typically, the weakly nonlinear equation includes the linear and nonlinear terms as

$$\frac{\mathrm{d}A}{\mathrm{d}t} = sA - \frac{1}{2}L|A|^2 A, \qquad s = \sigma + i\omega = \sigma_1 + i\omega_1, \qquad L = L_r + iL_i$$

L is the Landau constant which is PDE specific. The Landau equation describes the magnitude

$$\frac{\mathrm{d}|A|^2}{\mathrm{d}t} = 2\sigma|A|^2 - L_r|A|^4$$

The $L_r|A|^4$ nonlinear term can either accelerate or saturate the growth depending on its sign. The complementary equation for the phase angle is

$$\frac{\mathrm{d}|\arg A|}{\mathrm{d}t} = \omega - \frac{1}{2}L_i|A|^2$$

The Landau equation can be written in the linear form, with its solution obtained as

$$\frac{\mathrm{d}|A|^{-2}}{\mathrm{d}t} + 2\sigma|A|^{-2} = L_r$$

$$|A|^2 = \frac{A_o^2}{\frac{L_r}{2\sigma}A_o^2 + \left(1 - \frac{L}{2\sigma}A_o^2\right)e^{-2\sigma t}}$$

Take $L_r > 0$ and $R > R_c$, we have $\sigma = \sigma_1 > 0$ and the magnitude $|A|$ behaves as

$$|A| \sim A_o e^{\sigma t}, \qquad t \to -\infty, \qquad |A| \to \sqrt{2\sigma/L_r} = \sqrt{2k(R - R_c)/L_r}, \qquad t \to +\infty$$

The disturbance equilibrates to a new base flow. It is steady if $\omega = 0$, and periodic if $\omega \neq 0$.

➢ Further examples

Viscous rotating / centrifugal flow instability

In the inviscid analysis, a parallel to the Rayleigh criterion can be formally made. There are three possible scenarios:

- Concentric circles with radius $R_1 < R_2$, rotating at Ω_1 and Ω_2 (Taylor-Couette flow)
- Flow in a curved pipe / channel with curvatures $R_1 < R_2$ (Dean's problem)
- Boundary layer on a curved wall with curvature R_o (Görtler's problem)

For the curved pipe, perturbation equations are written in the cylindrical coordinates. Assume a narrow gap d between two walls, and the Dean number Λ is defined as

$$\Lambda = 36 \frac{V_{\text{avg}}}{r} \frac{d}{\bar{R}}, \qquad \bar{R} = \frac{R_1 + R_2}{2}$$

V_{avg} is the bulk velocity, and the first term is effectively the Reynolds number.

For the boundary layer on a curved wall, assume the boundary layer thickness $\delta \ll R_o$. We can neglect the centrifugal effects in the base flow, i.e. assume that the boundary layer solution is parallel to the wall, but partially retain them in the disturbance equations. The Görtler number G is defined as

$$G = \frac{U_\infty \theta}{r} \sqrt{\frac{\theta}{R_o}}$$

θ is the momentum thickness of the boundary layer. An exchange of instability is observed (as in the Dean's problem), and the observed structure is the streamwise aligned Görtler vortices.

Compressible parallel shear flow

Relative to the incompressible case, the compressible one is complicated as we need to consider temperature T (or enthalpy h) and density ρ are also variables. Also, we have the Mach number $M = \bar{u}/a$ with sound speed a entering the problem. The base flow, density and temperature profiles are $U(y)$, $\rho(y)$ and $T(y)$. Analysis proceeds similarly to the incompressible case. In the compressible case, perturbations may travel supersonically relative to the base flow, which requires additional analysis.