ME 340 Mechanics: Elasticity & Inelasticity

Instructor: Wei Cai

Topics to be covered:

1. Fundamental variables and equations of elasticity
   - Stress, strain, tensor transformation
   - Stress-strain behavior of materials, Hooke's law
   - Equilibrium and compatibility conditions, Boundary value problem
2. Stress function method for 2D problems / Green's function in 3D
   - Plane strain and plane stress formulations, Airy stress function
   - Polynomial solution of beam, Weak/Strong boundary condition
   - Fourier method of solution, Elastic half space, Contact problem (2D/3D)
   - Elasticity in polar coordinates, Void, Pressurized tube, Wedge
   - Crack dislocation, Line force loading
3. Plasticity
   - Fundamental equations and their graphical representation
   - Beam bending, internal stress, rod torsion, pressure vessel
   - Hardening laws, plastic instability
   - Finite element model for plasticity
4. Fracture
   - Linear elastic fracture mechanics (LEFM)
   - Stress intensity factor, energy release rate, J-integral
   - Plastic zone in ductile fracture, Dugdale-Barrenblatt model
   - Cohesive zone model, Finite element model for fracture
   - Microscopic mechanisms of plasticity and fracture
   - Fatigue crack initiation and growth

Textbooks:

- J. R. Barber, *Elasticity*, 3rd edition
- T. L. Anderson, *Fracture Mechanics*, 3rd edition

Lecture 1. Stress, Strain, Elasticity

➢ Vectors & Vector transformation

Vectors have **magnitudes** and **directions**. Under a specific coordinate $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, we have one representation u_i of the vector $\boldsymbol{u}$. We also have another coordinate $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ with another representation u'_i. We have

$$\boldsymbol{u} = u_i \boldsymbol{e}_i = u'_i \boldsymbol{e}'_i, \qquad u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad u'_i = \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix}$$

The transformation matrix Q is orthogonal with $Q^{-1} = Q^T$ and is defined as

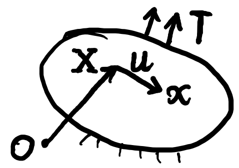
$$Q_{ij} = \langle \boldsymbol{e}'_i, \boldsymbol{e}_j \rangle, \qquad u'_i = Q_{ij} u_j, \qquad \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

➢ Displacement, Strain, Stress, Generalized Hooke's law

For **linear infinitesimal** elasticity, the displacement and strain are

$$\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{x} - \boldsymbol{X}$$

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$$



The strain tensor is symmetric. The rotation tensor does not contribute to deformation

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}), \qquad \varepsilon_{ij} + \omega_{ij} = u_{i,j}$$

Given the stress cube, the stress tensor is symmetric and is defined as

$$\sigma_{ij} = \sigma_{ji} = \frac{\text{Force in j-th direction}}{\text{Unit area in i-th face}}$$

The traction becomes

$$T_j = \sigma_{ij} n_i$$

The generalized Hooke's law is

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \qquad \varepsilon_{ij} = S_{ijkl} \sigma_{kl}$$

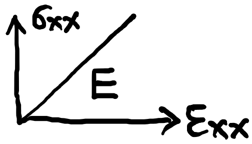
➢ Tensor transformation

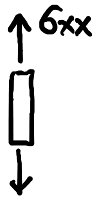
$$\varepsilon'_{ij} = Q_{ip} Q_{jq} \varepsilon_{pq}, \qquad \varepsilon' = Q \varepsilon Q^T$$

Lecture 2. Anisotropy, Equation for elasticity

➢ Anisotropic & Isotropic elasticity

For a tensile test, what we really describe is the relationship between σ_{xx} and ε_{xx}





Voigt notation

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} \rightarrow \begin{bmatrix}\sigma_{1}\\ \sigma_{2}\\ \sigma_{3}\\ \sigma_{4}\\ \sigma_{5}\\ \sigma_{6}\end{bmatrix}, \quad \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{bmatrix} = \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} \rightarrow \begin{bmatrix}\varepsilon_{1}\\ \varepsilon_{2}\\ \varepsilon_{3}\\ \varepsilon_{4}\\ \varepsilon_{5}\\ \varepsilon_{6}\end{bmatrix}, \quad \begin{bmatrix}\sigma_{1}\\ \sigma_{2}\\ \sigma_{3}\\ \sigma_{4}\\ \sigma_{5}\\ \sigma_{6}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ \cdot & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ \cdot & \cdot & C_{33} & C_{34} & C_{35} & C_{36}\\ \cdot & \cdot & \cdot & C_{44} & C_{45} & C_{46}\\ \cdot & \text{sym} & \cdot & \cdot & C_{55} & C_{56}\\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{1}\\ \varepsilon_{2}\\ \varepsilon_{3}\\ \varepsilon_{4}\\ \varepsilon_{5}\\ \varepsilon_{6}\end{bmatrix}$$

We can now write the Hooke's law as

$$\sigma_I = C_{IJ}\varepsilon_J, \qquad \varepsilon_I = S_{IJ}\sigma_J, \qquad I,J = 1,2,\cdots,6$$

However, to perform transformation, we still need to apply on the 4th order tensor C_{ijkl}

Independent components of tensors

	Components	Independent components	Symmetry
σ_{ij}	9	6	$\sigma_{ij} = \sigma_{ji}$
C_{ijkl}	81	21	$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$

The major symmetry $C_{IJ} = C_{JI}$ is due to the existence of a strain energy density that satisfies

$$\sigma_I = \frac{\partial w}{\partial \varepsilon_I}, \qquad C_{IJ} = C_{JI} = \frac{\partial w}{\partial \varepsilon_I \partial \varepsilon_J}$$

Isotropic material

$$S_{11} = S_{1111} = S_{22} = S_{33} = \frac{1}{E} \qquad \varepsilon_{11} = \frac{\sigma_{11}}{E}$$

$$S_{12} = S_{1122} = S_{13} = S_{23} = -\frac{\nu}{E} \qquad \varepsilon_{22} = -\nu\frac{\sigma_{11}}{E}$$

$$S_{44} = S_{55} = S_{66} = \frac{1}{G} = 2(S_{11} - S_{12}) \qquad \varepsilon_{23} = \frac{1}{2}\gamma_{23} = \frac{\sigma_{23}}{2G}, \qquad E = 2(1+\nu)G$$

For **isotropic elasticity**, the Hooke's law becomes

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad \varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$

- Equation for elasticity

Compatibility condition

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

This condition is automatically satisfied if the strain is obtained from the displacement.

Another perspective: ε_{ij} has 6 degrees of freedom, while u_i has 3 degrees of freedom. We require the compatibility condition to make sure we can find u_i that corresponds to ε_{ij} (Implications: No cracks, gaps, discontinuities, etc.)

Equilibrium condition (static)

$$\sigma_{ij,i} + F_j = 0, \qquad \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{F} = 0$$

- General strategies for solution

1. (**3D problem**) Start from u_i and use the Hooke's law

For isotropic elasticity, the stress σ_{ij} can be written as

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$$

Then, using the equilibrium condition to write PDE for u_i

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} + F_i = 0$$

In vector form, we have

$$\mu\nabla^2\boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{F} = \boldsymbol{0}$$

2. (**2D problem**) Write out the compatibility condition in terms of stress

The compatibility condition in **2D plane strain** becomes one single equation

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0$$

The equilibrium condition gives

$$\sigma_{xx,x} + \sigma_{yx,y} + F_x = 0$$
$$\sigma_{xy,x} + \sigma_{yy,y} + F_y = 0$$

We begin with a trial solution / ansatz $\phi(x, y)$ with

$$\sigma_{xx} = \phi_{,yy} + V \qquad \sigma_{yy} = \phi_{,xx} + V \qquad \sigma_{xy} = -\phi_{,xy}, \qquad F_x = -V_{,x}, \qquad F_y = -V_{,y}$$

The equilibrium condition is automatically satisfied. Now the compatibility condition gives

$$\nabla^2(\nabla^2\phi) = \nabla^4\phi = 0$$

CA Session 1. Euler-Bernoulli beam theory, Stiffness tensor

➢ Euler-Bernoulli beam theory

Assumptions

1. Plane strain
2. Newton axis (N.A.)
3. Small deformation
4. Plane surfaces are perpendicular to the Newton axis

Flexural stiffness

The curvature is related to the bending moment divided by the flexural stiffness

$$K = \frac{d\theta}{dx} = \frac{d^2 w}{dx^2} = \frac{M}{EI_z}$$

The moment of inertia I_z and the flexural stiffness are calculated as

$$I_z = \int_{-b}^{b} y^2 t dy, \qquad EI_z = \frac{2Etb^3}{3}$$

Force and angular moment balances

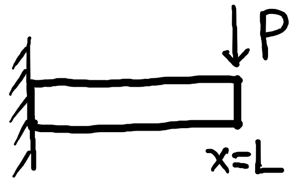
$$\frac{dV(x)}{dx} = -q(x), \qquad \frac{dM(x)}{dx} = V(x)$$

Beam equation

$$EI_z \frac{d^4 w(x)}{dx^4} = -q(x)$$

We need 4 boundary conditions in total

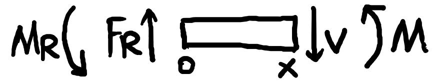
➢ Cantilever beam with load at the free end



Reaction force and angular moment are

$$|F_R| = P, \qquad |M_R| = PL$$

From shear force and bending moment balances, we have



$$V(x) = P, \qquad PL + M = xV(x), \qquad M = P(x - L)$$

The bending moment is contributed by normal stress σ_{xx}, which is

$$\sigma_{xx}(x,y) = -\frac{M(x)y}{I_z}, \qquad M(x) = \int_{-b}^{b} \sigma_{xx}(x,y) ty \, \mathrm{d}y$$

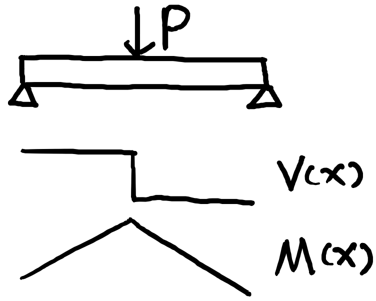
The shear force is contributed by the shear stress σ_{xy}, which is

$$\sigma_{xy}(x,y) = \frac{3V(x)}{2A}\left[1 - \left(\frac{y}{b}\right)^2\right], \qquad V(x) = \int_{-b}^{b} \sigma_{xy}(x,y) t \, \mathrm{d}y$$

- Beam supported at both ends with a central load

At the left half, reaction force at the pin support is

$$V(0) = \frac{P}{2}$$



From shear force and bending moment balances, we have

$$V(x) = \frac{P}{2}, \qquad M(x) = \frac{Px}{2}, \qquad 0 < x < \frac{L}{2}$$

For the right half, we similarly have

$$V(x) = -\frac{P}{2}, \qquad M(x) = \frac{P}{2}(L - x), \qquad \frac{L}{2} < x < L$$

- Cantilever beam with a linear load

Reaction force and angular moment are

$$q(x) = \frac{kx}{L}, \qquad |F_R| = \int_0^L q(x)dx = \frac{kL}{2}, \qquad |M_R| = \int_0^L xq(x)\,\mathrm{d}x = \frac{kL^2}{3}$$

From shear force and bending moment balances, we have

$$V(x) = |F_R| - \int_0^x q(x)\,\mathrm{d}x\,, \qquad M(x) = xV(x) + \int_0^x xq(x)\,\mathrm{d}x - |M_R|$$

Directly from the ODEs, we have

$$\frac{dV}{dx} = -\frac{kx}{L}, \qquad V(L) = 0, \qquad V(x) = \frac{k}{2L}(L^2 - x^2)$$

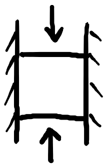
$$\frac{dM}{dx} = \frac{k}{2L}(L^2 - x^2), \qquad M(L) = 0, \qquad M(x) = \frac{kLx}{2} - \frac{kx^3}{6L} - \frac{kL^2}{3}$$

Lecture 3. 2D Elasticity, Airy stress function

- Types of 2D elasticity problem

Antiplane shear: Only $u_z(x, y)$

Plane strain



$$u_x(x,y), \qquad u_y(x,y), \qquad u_z = 0, \qquad \frac{\partial}{\partial z} = 0$$

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0, \qquad \sigma_{xz} = \sigma_{yz} = 0, \qquad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \neq 0$$

Stress-strain relations

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}$$

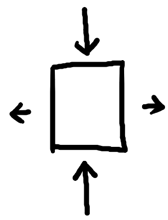
$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} = -\frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}$$

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

We can define the effective E' and ν'

$$\frac{1}{E'} = \frac{1-\nu^2}{E}, \qquad \nu' = \frac{\nu}{1-\nu}, \qquad \frac{\nu'}{E'} = \frac{\nu(1+\nu)}{E}$$

Plane stress



The material is free to expand in z-direction

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0, \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0, \qquad \varepsilon_{zz} \neq 0$$

Stress-strain relations

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}, \qquad \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy}, \qquad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

However, for plane stress the compatibility condition for ε_{zz} also needs to be satisfied.

If the material expands in z-direction, there will be stress dependence on z-direction, since the material elements are connected, which breaks down the 2D plane stress assumption.

Kolosov's constant

$$\kappa = 3 - 4\nu, \qquad \kappa = \frac{3-\nu}{1+\nu}$$

- Airy stress function

For **plane strain** **without body force**, we begin with a trial solution / ansatz $\phi(x, y)$ with

$$\sigma_{xx} = \phi_{,yy} \qquad \sigma_{yy} = \phi_{,xx} \qquad \sigma_{xy} = -\phi_{,xy}$$

The equilibrium condition is automatically satisfied.

The compatibility condition becomes the bi-harmonic equation

$$\nabla^2(\nabla^2\phi) = \nabla^4\phi = 0$$

Plane stress shares the same equation, but an extra compatibility condition for ε_{zz} needs to be considered (but usually ignored)

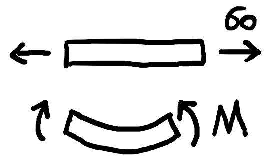
Examples

$$\phi(x,y) = \alpha x + \beta y + \gamma \qquad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$$

$$\phi(x,y) = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 - Cxy \qquad \sigma_{xx} = B, \;\; \sigma_{yy} = A, \;\; \sigma_{xy} = C$$

$$\phi(x,y) = \frac{1}{2}\sigma_0 y^2 \qquad \sigma_{xx} = \sigma_0$$

$$\phi(x,y) = -\frac{M}{6I_z}y^3 \qquad \sigma_{xx} = -\frac{My}{I_z}$$

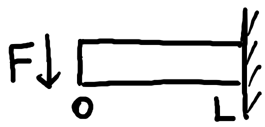


- Rectangular beam: Boundary value problem (BVP)

Boundary conditions

Top & Bottom (Traction free): Strong boundary conditions

$$T_x = \sigma_{xy} = 0, \qquad T_y = \sigma_{yy} = 0, \qquad at\;\; y = \pm b$$



Left side: Weak boundary conditions (consider thickness $t = 1$)

$$\int_{-b}^{b} \sigma_{xy}\, \mathrm{d}y = F, \qquad \int_{-b}^{b} \sigma_{xx}\, \mathrm{d}y = 0, \qquad \int_{-b}^{b} \sigma_{xx} y\, \mathrm{d}y = 0, \qquad at\;\; x = 0$$

Right side: Weak boundary conditions for stress are automatically satisfied since the equilibrium condition is satisfied at every point during the solution.

However, the strong boundary conditions for displacement $u_x = u_y = 0$ at $x = L$ still need to be considered after obtaining the strain field. But in practice, only the weak conditions for displacement can be satisfied by the solution.

Solution

In this problem the moment $M \propto x$, so we first choose

$$\phi(x,y) = C_1 xy^3, \qquad \sigma_{xx} = 6C_1 xy, \qquad \sigma_{yy} = 0, \qquad \sigma_{xy} = -3C_1 y^2$$

However, to satisfy $\sigma_{xy} = 0$ at $y = \pm b$, we need to fix this component as

$$\sigma_{xy} = -3C_1 y^2 + 3C_1 b^2, \qquad \phi = C_1 xy^3 - 3C_1 b^2 xy, \qquad \nabla^4\phi = 0$$

The other stress components are unchanged.

The weak boundary conditions at $x = 0$ gives

$$\int_{-b}^{b} \sigma_{xy}\, \mathrm{d}y = 3C_1 \int_{-b}^{b} (b^2 - y^2)\, \mathrm{d}y = 4b^3 C_1 = F, \qquad C_1 = \frac{F}{4b^3}$$

The stress field solutions become

$$\sigma_{xx} = \frac{3F}{2b^3} xy, \qquad \sigma_{xy} = \frac{3F}{4b^3}(b^2 - y^2), \qquad \sigma_{yy} = 0$$

We can obtain the strain fields from the Hooke's law (under plane stress)

$$\varepsilon_{xx} = \frac{3F}{2Eb^3} xy, \qquad \varepsilon_{yy} = -\frac{3F\nu}{2Eb^3} xy, \qquad \varepsilon_{xy} = \frac{3F(1+\nu)}{4Eb^3}(b^2 - y^2)$$

Integration leads to the displacement fields

$$u_x = \frac{3F}{4Eb^3} x^2 y + f(y), \qquad u_y = -\frac{3F\nu}{4Eb^3} xy^2 + g(x)$$

To satisfy ε_{xy} we need

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \frac{3F(1+\nu)}{2Eb^3}(b^2 - y^2) = \frac{3F}{4Eb^3} x^2 + f'(y) - \frac{3F\nu}{4Eb^3} y^2 + g'(x)$$

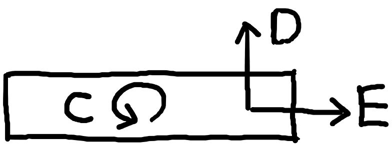
Because the equality holds for any x and y, separation of variables leads to

$$\frac{3F}{4Eb^3} x^2 + g'(x) = \frac{3F(1+\nu)}{2Eb^3}(b^2 - y^2) + \frac{3F\nu}{4Eb^3} y^2 - f'(y) = C$$

We finally have

$$g(x) = -\frac{F}{4Eb^3} x^3 + Cx + D, \quad f(y) = \frac{F(1+\nu)}{2Eb^3}(3b^2 y - y^3) + \frac{F\nu}{4Eb^3} y^3 - Cy + E$$

The constants D and E denote rigid body translation, while C denotes rotation



Lecture 4. Saint-Venant's principle, General solution in rectangular domain

- Saint-Venant's principle

Saint-Venant's principle justifies the usage of weak boundary conditions.

It states that the stress field in a rod or beam sufficiently far away from its end produced by some traction forces applied to the end is independent of the detailed distribution of the traction on the cross section, as long as the total force and total moment applied to the end surface remain the same.

In other words, the correction term is localized (decays exponentially), and the characteristic length scale is about the beam height $2b$.

We also want to apply strong boundary conditions to the longer dimension.

- General solution in rectangular domain

From Euler-Bernoulli beam theory, for polynomial loads we have

$$q(x) \sim x^n, \qquad V(x) \sim x^{n+1}, \qquad M(x) \sim x^{n+2}$$

Then we can suppose the stress function ϕ has a maximum order of $n+5$

$$\sigma_{xx} \sim x^{n+2}y, \qquad \phi \sim x^{n+2}y^3$$

All we need to do is solve the coefficients of the polynomial, using the biharmonic equation and boundary conditions.

Lecture 5. Fourier solution

➢ Independent solutions of biharmonic equation

For a rectangular domain $[-a,a]\times[-b,b]$, we describe the strong boundary conditions

$$\sigma_{yy}(x,y=b)=t_{y+}(x),\qquad \sigma_{yy}(x,y=-b)=-t_{y-}(x)$$
$$\sigma_{xy}(x,y=b)=t_{x+}(x),\qquad \sigma_{xy}(x,y=-b)=-t_{x-}(x)$$

Consider the trial solution

$$\phi(x,y)=e^{\alpha x}e^{\beta y}$$

If the harmonic equation $\nabla^2\phi=0$ is satisfied, then we have

$$\nabla^2\phi=(\alpha^2+\beta^2)\phi=0,\qquad \alpha^2+\beta^2=0$$

Then the stress function has the form

$$\phi(x,y)=e^{\pm i\lambda x}e^{\pm\lambda y},\qquad \lambda\in\mathbb{R}$$

Now for the biharmonic equation $\nabla^4\phi=0$, the four independent solutions are

$$e^{i\lambda x}e^{\lambda y},\qquad e^{i\lambda x}e^{-\lambda y},\qquad e^{i\lambda x}\cdot ye^{\lambda y},\qquad e^{i\lambda x}\cdot ye^{-\lambda y}$$

Then the stress function has the **general form**

$$\phi(x,y)=e^{i\lambda x}\left[(c_1+c_2y)e^{\lambda y}+(c_3+c_4y)e^{-\lambda y}\right]$$

Symmetric solutions

The general form can be decomposed into even and odd functions

$$\phi(x,y)=\begin{pmatrix}\cos\lambda x\\ \sin\lambda x\end{pmatrix}\times\begin{pmatrix}A'\cosh\lambda y+D'y\sinh\lambda y\\ B'y\cosh\lambda y+C'\sinh\lambda y\end{pmatrix}$$

➢ Sinusoidal loading on the rectangular beam

Consider the rectangular beam with loading

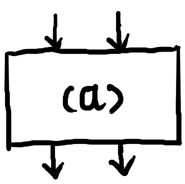
$$p_y(x)=p_0\cos\left(\frac{\pi x}{2a}\right)$$

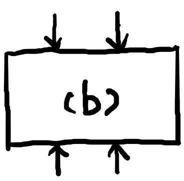
The two ends are simply supported, both sides having equal forces to balance the loading.

The strong boundary conditions are

$$\sigma_{yy}(x,y=b)=-p_0\cos\left(\frac{\pi x}{2a}\right),\qquad \sigma_{xy}(x,y=b)=0$$
$$\sigma_{yy}(x,y=-b)=0,\qquad \sigma_{xy}(x,y=-b)=0$$

We can decompose the problem into two simple scenarios





Problem (a): σ_{yy} even in x, odd in y

$$\sigma_{yy}(x, y = b) = -\frac{p_0}{2}\cos\left(\frac{\pi x}{2a}\right), \qquad \sigma_{yy}(x, y = -b) = \frac{p_0}{2}\cos\left(\frac{\pi x}{2a}\right)$$

Problem (b): σ_{yy} even in x, even in y

$$\sigma_{yy}(x, y = b) = -\frac{p_0}{2}\cos\left(\frac{\pi x}{2a}\right), \qquad \sigma_{yy}(x, y = -b) = -\frac{p_0}{2}\cos\left(\frac{\pi x}{2a}\right)$$

For problem (a), we have

$$\phi(x, y) = \cos\lambda x \cdot (By\cosh\lambda y + C\sinh\lambda y), \qquad \lambda = \frac{\pi}{2a}$$

The top boundary conditions give

$$\sigma_{yy}(y = b) = -\lambda^2\cos\lambda x \cdot (Bb\cosh\lambda b + C\sinh\lambda b) = -\frac{p_0}{2}\cos\lambda x$$

$$\sigma_{xy}(y = b) = \lambda\sin\lambda x \cdot [\lambda Bb\sinh\lambda b + (B + \lambda C)\cosh\lambda b] = 0$$

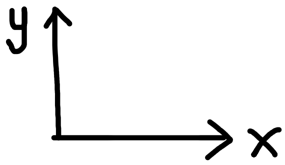
Lecture 6. Elastic halfspace

- Issues to solve halfspace problem

  Euler-Bernoulli beam theory: Proportional to $1/I$, for halfspace all fields go to 0

  Airy stress function: Solution is very tedious, and we only focus on displacement at the top. However, we need to use the general procedure to solve the Green's function

- **2D** elastic halfspace problem



Find displacement at x due to a force at x' using the Green's function

$$u_i(x) = \int_\Omega G_{ij}^S(x - x')T_j(x')\,\mathrm{d}x', \qquad i,j = x,y$$

Single Fourier mode

For a sinusoidal loading with wavenumber k, we have

$$T_y(x) = e^{ikx}, \qquad u_y(x) = \int_{-\infty}^{+\infty} G_S(x - x')e^{ikx'}\,\mathrm{d}x'$$

A change of variable leads to

$$u_y(x) = e^{ikx}\int_{-\infty}^{+\infty} G_S(x')e^{-ikx'}\,\mathrm{d}x' = \hat{G}_S(k)e^{ikx} = \hat{G}_S(k)T_y(x)$$

For a single Fourier mode k, displacement is proportional to surface loading. The factor is the Fourier domain Green's function at wavenumber k.

Green's function for a normal loading

Consider the surface load

$$T_y(x) = T_0 \cos kx$$

Surface boundary conditions

$$\sigma_{yy}(x, y = 0) = T_y(x), \qquad \sigma_{xy}(x, y = 0) = 0$$

We choose the Airy stress function that converges at $y \to -\infty$

$$\phi(x,y) = \cos kx\,(A + By)e^{ky}$$

Stress fields become

$$\sigma_{yy} = -k^2 \cos kx\,(A + By)e^{ky}, \qquad \sigma_{xy} = k \sin kx\,(kA + B + Bky)e^{ky}$$

The boundary conditions give

$$-k^2 A = T_0, \qquad k^2 A + kB = 0, \qquad A = -\frac{T_0}{k^2}, \qquad B = \frac{T_0}{k}$$

Finally, the solutions are

$$\sigma_{xx} = T_0 \cos kx \cdot (1 + ky)e^{ky}$$

$$\sigma_{yy} = T_0 \cos kx \cdot (1 - ky)e^{ky}$$

$$\sigma_{xy} = T_0 \sin kx \cdot kye^{ky}$$

Under plane strain assumption, we can further obtain the strain and displacement fields

$$u_x(x,y) = \frac{T_0}{kE} \sin kx \left[(1 - \nu - 2\nu^2) + (1+\nu)ky\right]e^{ky} + C$$

$$u_y(x,y) = \frac{T_0}{kE} \cos kx \left[(2 - 2\nu^2) - (1+\nu)ky\right]e^{ky} + D$$

At the surface $y = 0$, the displacement fields are

$$u_x(x, y = 0) = \tilde{u}_x(x) = \frac{T_0}{kE} \sin kx \cdot (1 - \nu - 2\nu^2)$$

$$u_y(x, y = 0) = \tilde{u}_y(x) = \frac{T_0}{kE} \cos kx \cdot (2 - 2\nu^2)$$

We can identify the Green's function G_{yy}^S as the following

$$G_{yy}^S(k) = \frac{2(1-\nu^2)}{|k|E} = \frac{1-\nu}{|k|\mu}$$

In the spatial domain, the inverse FT results in

$$G_{yy}^S(x) = -\frac{1-\nu}{\pi\mu}\ln|x| = -\frac{\kappa+1}{4\pi\mu}\ln|x|$$

For displacement u_x, we can identify a phase shift in the Green's function

$$G_{xy}^S(k) = -i\frac{1-\nu-2\nu^2}{kE} = -i\frac{1-2\nu}{2k\mu}$$

In the spatial domain, we have

$$G_{xy}^S(x) = \frac{1-2\nu}{4\mu}\operatorname{sgn}(x) = \frac{\kappa-1}{8\mu}\operatorname{sgn}(x)$$

- **Green's function for 2D elastic halfspace**

y-loading	G_{yy}^S	$\frac{\kappa+1}{4\mu}\cdot\frac{1}{\lvert k\rvert}$	$-\frac{\kappa+1}{4\pi\mu}\ln\lvert x\rvert$
	G_{xy}^S	$-\frac{\kappa-1}{4\mu}\cdot\frac{i}{k}$	$\frac{\kappa-1}{8\mu}\operatorname{sgn}(x)$
x-loading	G_{xx}^S	$\frac{\kappa+1}{4\mu}\cdot\frac{1}{\lvert k\rvert}$	$-\frac{\kappa+1}{4\pi\mu}\ln\lvert x\rvert$
	G_{yx}^S	$\frac{\kappa-1}{4\mu}\cdot\frac{i}{k}$	$-\frac{\kappa-1}{8\mu}\operatorname{sgn}(x)$

Note: This matrix is anti-symmetric

Lecture 7. Polar coordinates

- Elasticity equations in polar coordinates

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}$$

Gradient operator

$$\boldsymbol{\nabla} = \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}$$

Derivatives of unit vectors

$$\frac{\partial\boldsymbol{e}_r}{\partial\theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial\boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r, \qquad \frac{\partial\boldsymbol{e}_r}{\partial r} = \frac{\partial\boldsymbol{e}_\theta}{\partial r} = 0$$

In tensor form, the stress tensor can be written as

$$\boldsymbol{\sigma} = (\boldsymbol{\nabla}^a \otimes \boldsymbol{\nabla}^a)\phi, \qquad \boldsymbol{\nabla}^a = \boldsymbol{\nabla}\times\boldsymbol{e}_z$$

The new gradient operator is

$$\boldsymbol{\nabla}^a = -\boldsymbol{e}_y\frac{\partial}{\partial x} + \boldsymbol{e}_x\frac{\partial}{\partial y} = -\boldsymbol{e}_\theta\frac{\partial}{\partial r} + \boldsymbol{e}_r\frac{1}{r}\frac{\partial}{\partial\theta}$$

Stress field from the stress function $\phi(r,\theta)$ now becomes

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \qquad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

Biharmonic equation

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0$$

In tensor form, the strain tensor can be written as

$$\boldsymbol{\varepsilon} = \frac{1}{2}[(\boldsymbol{\nabla}\otimes\boldsymbol{u}) + (\boldsymbol{\nabla}\otimes\boldsymbol{u})^T]$$

Strain field from displacement

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)$$

Generalized Hooke's law

$$\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon}$$

Traction force

$$\boldsymbol{T} = \boldsymbol{n}\cdot\boldsymbol{\sigma}, \qquad T_r = \sigma_{rr}n_r + \sigma_{\theta r}n_\theta, \qquad T_\theta = \sigma_{r\theta}n_r + \sigma_{\theta\theta}n_\theta$$

Equilibrium condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{F} = \boldsymbol{0}$$

- **Michell solutions** to biharmonic equation in polar coordinates

We expect a periodic function in θ

$$\phi(r,\theta)=\phi(r,\theta+2\pi)$$

Consider the form of stress function

$$\phi(r,\theta)=f(r)\,e^{in\theta},\qquad n=0,1,2,\cdots$$

The biharmonic equation becomes

$$\nabla^4\phi=0\quad\Leftrightarrow\quad\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}\right)f(r)=0$$

The polynomial form $f(r)=r^m$ gives

$$(m^2-n^2)[(m-2)^2-n^2]\,r^{m-4}=0$$

This indicates the general **Michell solutions**

$$\phi(r,\theta)=(A_{n1}r^{n+2}+A_{n2}r^{-n+2}+A_{r3}r^{n}+A_{r4}r^{-n})e^{in\theta},\qquad n=2,3,4,\cdots$$

For $n=0$, we have

$$f_0(r)=A_{01}r^2+A_{02}r^2\ln r+A_{03}\ln r+A_{04}\theta$$

For $n=1$, we have

$$f_1(r)=A_{11}r^3+A_{12}r\ln r+A_{13}r\theta+\frac{A_{14}}{r}$$

- Example: Shear loading, circular hole in a plate

Infinity conditions

$$\sigma_{xx}=\sigma_{yy}=0,\qquad \sigma_{xy}=S,\qquad r\to\infty$$

Zero normal traction at the hole

$$\sigma_{r\theta}=\sigma_{rr}=0,\qquad r=a$$

We decompose the stress function into two parts

$$\phi=\phi^{(0)}+\phi^{(1)}$$

The uniform shear loading is described by

$$\phi^{(0)}=-Sxy=-Sr^2\sin\theta\cos\theta=-\frac{1}{2}Sr^2\sin 2\theta\,,\qquad \sigma_{xy}^{(0)}=S$$

$$\sigma_{rr}^{(0)}=S\sin 2\theta\,,\qquad \sigma_{\theta\theta}^{(0)}=-S\sin 2\theta\,,\qquad \sigma_{r\theta}^{(0)}=S\cos 2\theta$$

Now we need to cancel the stress at $r=a$. The effect of the circular hole is described by

$$\phi^{(1)}=f_2(r)\sin 2\theta=(A+Br^{-2})\sin 2\theta$$

$$\sigma_{rr}^{(1)}=-\left(\frac{4A}{r^2}+\frac{6B}{r^4}\right)\sin 2\theta\,,\qquad \sigma_{r\theta}^{(1)}=\left(\frac{2A}{r^2}+\frac{6B}{r^4}\right)\cos 2\theta$$

At $r=a$, the boundary conditions give

$$\frac{4A}{a^2}+\frac{6B}{a^4}=S,\qquad \frac{2A}{a^2}+\frac{6B}{a^4}=-S,\qquad A=Sa^2,\qquad B=-\frac{1}{2}Sa^4$$

The stress fields are obtained as

$$\sigma_{rr} = S\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\sin 2\theta$$

$$\sigma_{r\theta} = S\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = -S\left(1 + \frac{3a^4}{r^4}\right)\sin 2\theta$$

The maximum shear stress at the hole $r = a$ is

$$\tau(\theta) = \sqrt{\left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2} = \frac{1}{2}|\sigma_{\theta\theta}| = 2S\sin 2\theta$$

Therefore, the stress-concentration factor is 2, as we have $\tau_{max} = 2S$

Lecture 8. Polar coordinates

- Example: Tensile loading, circular hole in a plate

Infinity conditions

$$\sigma_{xy} = \sigma_{yy} = 0, \qquad \sigma_{xx} = S, \qquad r \to \infty$$

The uniform tensile loading in x-direction is described by

$$\phi^{(0)} = \frac{1}{2}Sy^2 = \frac{1}{2}Sr^2 \sin^2\theta = \frac{1}{4}Sr^2(1-\cos 2\theta), \qquad \sigma_{xx}^{(0)} = S$$

The effect of the circular hole is described by $n = 0$ and $n = 2$ solutions

$$\phi^{(1)} = f_0(r) + f_2(r)\sin 2\theta = A\ln r + B\theta + (C + Dr^{-2})\cos 2\theta$$

At $r = a$, the boundary conditions give

$$A = -\frac{Sa^2}{2}, \qquad B = 0, \qquad C = \frac{Sa^2}{2}, \qquad D = -\frac{Sa^2}{4}$$

The stress fields are obtained as

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{S}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

The maximum normal stress at the hole $r = a$ is

$$\sigma(\theta) = |\sigma_{\theta\theta}| = S - 2S\cos 2\theta$$

Therefore, the stress-concentration factor is 3, as we have $\sigma_{max} = 3S$

- Example: Rotated tensile loading, circular hole in a plate

For anti-clockwise and clockwise rotation, the new angle becomes

$$\theta' = \theta \pm \frac{\pi}{4}, \qquad 2\theta' = 2\theta \pm \frac{\pi}{2}$$

Therefore, the solution of the rotated tensile loading is

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) \pm \frac{S}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\sin 2\theta$$

$$\sigma_{r\theta} = \mp\frac{S}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^@}\right) \pm \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\sin 2\theta$$

We add these two scenarios together, and obtain the biaxial tensile loading case

$$\sigma_{rr} = S\left(1 - \frac{a^2}{r^2}\right), \qquad \sigma_{r\theta} = 0, \qquad \sigma_{\theta\theta} = S\left(1 + \frac{a^2}{r^2}\right)$$

We subtract these two scenarios, and obtain the shear loading case

- Example: Infinite pressure vessel

The problem is decomposed into: Compression + Biaxial tensile loading

$$\sigma_{rr} = -p_0 + \ p_0\left(1 - \frac{a^2}{r^2}\right) = -p_0\frac{a^2}{r^2}, \qquad \sigma_{\theta\theta} = -p_0 + p_0\left(1 + \frac{a^2}{r^2}\right) = p_0\frac{a^2}{r^2}, \qquad \sigma_{r\theta} = 0$$

- Example: Thick-walled pressure vessel

The trial solution can be constructed from the infinite pressure vessel solution

$$\sigma_{rr} = A - \frac{B}{r^2}, \qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}, \qquad \sigma_{r\theta} = 0$$

The boundary conditions at $r = r_1$ and $r = r_2$ give

$$\sigma_{rr}(r_1) = A - \frac{B}{r_1^2} = -p_1, \qquad \sigma_{rr}(r_2) = A - \frac{B}{r_2^2} = -p_2$$

We eventually have the solution

$$A = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}, \qquad B = \frac{p_2 - p_1}{\frac{1}{r_1^2} - \frac{1}{r_2^2}}$$

When $p_2 = 0$, the solution becomes

$$\sigma_{rr} = \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left(1 - \frac{r_2^2}{r^2}\right), \qquad \sigma_{\theta\theta} = \frac{p_1 r_1^2}{r_2^2 - r_1^2}\left(1 + \frac{r_2^2}{r^2}\right)$$

In the thin wall limit, we have

$$r_2 = r_1 + t, \qquad r_2^2 - r_1^2 \approx 2r_1 t, \qquad t \ll r_1$$

The maximum stress then becomes

$$\sigma_{\theta\theta}(r = r_1) = p_1\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} \approx \frac{p_1 r_1}{t}$$

Lecture 9. Contact problem

➢ Displacement from arbitrary surface loading

$$\tilde{u}_x(x) = \int_{-\infty}^{+\infty} \left[G_{xx}^S(x-x')\,T_x(x') + G_{xy}^S(x-x')\,T_y(x')\right] \mathrm{d}x'$$

$$\tilde{u}_y(x) = \int_{-\infty}^{+\infty} \left[G_{yx}^S(x-x')\,T_x(x') + G_{yy}^S(x-x')\,T_y(x')\right] \mathrm{d}x'$$

➢ Frictionless contact problem: Formulation

We will mostly work with normal forces for contact problems

$$G_{yy}^S(x) = -\frac{\kappa+1}{4\pi\mu}\ln|x|$$

The compressive surface pressure loading is $p_y(x)$ and the vertical displacement is

$$\tilde{u}_y(x) = -\int_{-\infty}^{+\infty} G_{yy}^S(x-x')\,p_y(x')\,\mathrm{d}x' = \frac{\kappa+1}{4\pi\mu}\int_{-\infty}^{+\infty} p_y(x')\ln|x-x'|\,\mathrm{d}x'$$

Consider the indenter is rigid and its shape is $u_0(x)$. The contact region is $x \in [-c, c]$. After the contact, the indenter goes down by d. In the contact area, we know the displacement

$$\tilde{u}_y(x) = u_0(x) - d, \qquad -c < x < c$$

For infinitesimal elasticity, we consider the material at x match the shape $u_0(x)$ even with non-zero horizontal displacement. Within the contact, we have

$$u_0(x) - d = \frac{\kappa+1}{4\pi\mu}\int_{-c}^{c} p_y(x')\ln|x-x'|\,\mathrm{d}x'\,, \qquad -c < x < c$$

The total downward normal force is

$$F = \int_{-c}^{c} p_y(x')\,\mathrm{d}x'$$

The force F as a function of indentation d and contact region c is useful in practice

As a summary, the boundary conditions are

$$\tilde{u}_y(x) = u_0(x) - d, \qquad \sigma_{yy}(x) = -p_y(x) < 0, \qquad \sigma_{xy}(x) = 0, \qquad -c < x < c$$

$$\tilde{u}_y(x) < u_0(x) - d, \qquad \sigma_{yy}(x) = 0, \qquad \sigma_{xy}(x) = 0, \qquad |x| > c$$

The contact problem is essentially an inverse problem

- General solution approach

First taking the derivative of the integral equation

$$\frac{4\pi\mu}{\kappa+1}\cdot\frac{\mathrm{d}u_0(x)}{\mathrm{d}x}=\int_{-c}^{c}\frac{p_y(x')}{x-x'}\,\mathrm{d}x',\qquad -c<x<c$$

We have the following form of the (singular) integral equation

$$g(x)=\int_{-c}^{c}\frac{f(x')}{x-x'}\,\mathrm{d}x',\qquad -c<x<c$$

To properly define the integral equation, the Cauchy principal value is considered

$$\mathrm{P.V.}\left[\int_{-c}^{c}\frac{f(x')}{x-x'}\,\mathrm{d}x'\right]=\lim_{\varepsilon\to 0}\left[\int_{-c}^{x-\varepsilon}\frac{f(x')}{x-x'}\,\mathrm{d}x'+\int_{x+\varepsilon}^{c}\frac{f(x')}{x-x'}\,\mathrm{d}x'\right]$$

If the contact region is infinity, we can directly refer to the Hilbert transform. The contact problem is difficult due to the fact that we only know the displacement for some regions

The general solution of the integral equation is

$$f(x)=-\frac{1}{\pi^2\sqrt{c^2-x^2}}\int_{-c}^{c}\frac{\sqrt{c^2-x'^2}\,g(x')}{x-x'}\,\mathrm{d}x'+\frac{F}{\pi\sqrt{c^2-x^2}}$$

The second term corresponds to the flat punch solution

- Example: Flat punch

For a flat punch, we directly know the contact region, and the shape is flat with $u_0'(x)=0$

$$p_y(x)=\frac{F}{\pi\sqrt{c^2-x^2}}$$

There are singularities at the corners $x=\pm c$. Consider $x=c-r$, the singular behavior is

$$\sigma\sim\frac{1}{\sqrt{r}}$$

Now consider that two elastic half-spaces are glued within a region and pull them apart. This is a crack problem, and the stress singularity is the same as the flat punch problem

- Example: Cylindrical punch

We approximate the indenter shape as

$$u_0(x)=\frac{x^2}{2R},\qquad \frac{\mathrm{d}u_0(x)}{\mathrm{d}x}=\frac{x}{R}$$

Now the solution becomes

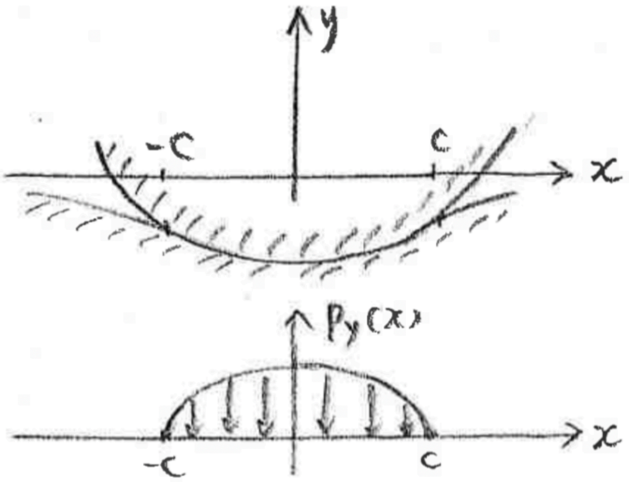
$$p_y(x)=\frac{4\mu}{(\kappa+1)R}\sqrt{c^2-x^2}+\left[\frac{F}{\pi}-\frac{2\mu c^2}{(\kappa+1)R}\right]\frac{1}{\sqrt{c^2-x^2}}$$

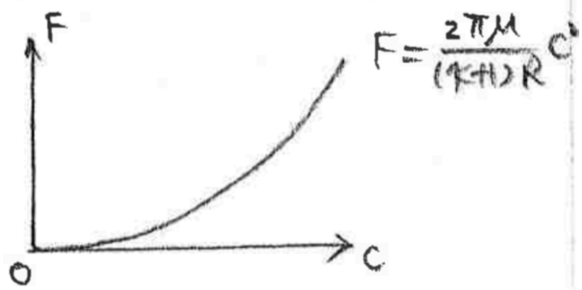
We notice the first term is finite, while the second term is singular. However, the contact region c is unknown, and we don't expect singularities to appear for the cylindrical punch. Therefore, we require

$$\frac{F}{\pi} - \frac{2\mu c^2}{(\kappa+1)R} = 0, \qquad c = \sqrt{\frac{(\kappa+1)R}{2\pi\mu}F}$$

The force distribution over the contact area becomes

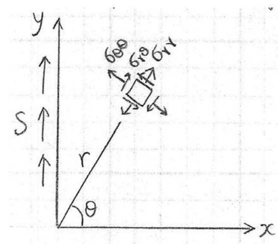
$$p_y(x) = \frac{4\mu}{(\kappa+1)R}\sqrt{c^2 - x^2} = \frac{2F}{\pi c^2}\sqrt{c^2 - x^2}$$





Lecture 10. Wedge and Notch

- Uniform shear on right-angle wedge



$$\sigma_{r\theta} = \sigma_{\theta\theta} = 0, \qquad \theta = 0$$

$$\sigma_{r\theta} = S, \qquad \sigma_{\theta\theta} = 0, \qquad \theta = \frac{\pi}{2}$$

We seek solution whose stress fields are independent of r, which requires $\phi \propto r^2$. We can find three terms from the Michell table, and the fourth one is $r^2\theta$. We therefore obtain

$$\phi = r^2(A_1 \cos 2\theta + A_2 + A_3 \sin 2\theta + A_4\theta)$$

The boundary conditions give

$$\phi = Sr^2\left(-\frac{\pi}{8}\cos 2\theta + \frac{\pi}{8} + \frac{\sin 2\theta}{4} - \frac{\theta}{2}\right)$$

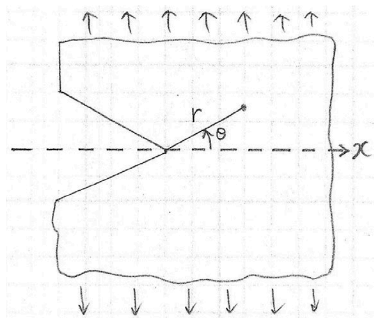
In Cartesian coordinate, we have

$$\phi = S\left[-\frac{\pi}{8}(x^2 - y^2) + \frac{\pi}{8}(x^2 + y^2) + \frac{xy}{2} + \frac{x^2 + y^2}{2}\arctan\frac{y}{x}\right]$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{Sy^2}{x^2 + y^2}$$

The stress is indeterminate at the corner $x = y = 0$. It is not a singular point, but the stress gradients in θ-direction increases with r^{-1} as $r \to 0$

- Notch: Re-entrant corner



The stress field should be singular at the corner. The boundary conditions are stated as

$$\sigma_{r\theta} = \sigma_{\theta\theta} = 0, \qquad \theta = \pm\alpha, \qquad \alpha > \frac{\pi}{2}$$

William's solution gives

$$\phi = r^{\lambda+1}[A_1 \cos(\lambda + 1)\theta + A_2 \cos(\lambda - 1)\theta + A_3 \sin(\lambda + 1)\theta + A_4 \sin(\lambda - 1)\theta]$$

We seek singular stress field solution with $\lambda < 1$. The boundary conditions lead to a linear system and to obtain non-trivial solutions, we need

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0, \qquad \det M_1 = 0 \quad \text{or} \quad \det M_2 = 0$$

We thus obtain the following equations

$$\lambda \sin 2\alpha + \sin 2\lambda\alpha = 0 \quad \text{or} \quad \lambda \sin 2\alpha - \sin 2\lambda\alpha = 0$$

With a change of variable, we have

$$x = 2\lambda\alpha, \qquad \frac{\sin 2\alpha}{2\alpha}x \pm \sin x = 0$$

- Semi-infinite crack

When $\alpha = \pi$, we have $\sin 2\pi\lambda = 0$ and the singular solution requires $\lambda = 0$ or $\lambda = 1/2$.

However, when $\lambda = 0$ the stress field $\sigma \propto r^{-1}$ and the strain energy is infinite

$$E_{el} = \frac{1}{2}\iint_S \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}S \propto \int_0^a \frac{1}{r}\cdot\frac{1}{r}\cdot r\mathrm{d}r = \int_0^a \frac{1}{r}\mathrm{d}r \to \infty$$

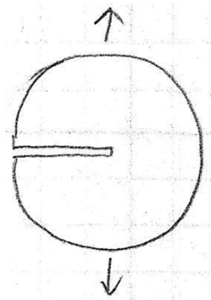
Therefore, the singular stress field for a crack tip is

$$\lambda = \frac{1}{2}, \qquad \sigma \sim \frac{1}{\sqrt{r}}$$

Mode I loading (symmetric solution)

$$A_1 = A(\lambda - 1)\sin(\lambda - 1)\alpha\,, \qquad A_2 = -A(\lambda + 1)\sin(\lambda + 1)\alpha$$

For semi-infinite crack, we have

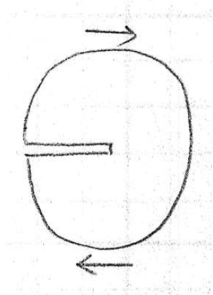


$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right), \qquad K_I = 3A\sqrt{\frac{\pi}{2}}$$
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$

Mode II loading (antisymmetric solution)

$$A_3 = A(\lambda + 1)\sin(\lambda - 1)\alpha\,, \qquad A_4 = -A(\lambda + 1)\sin(\lambda + 1)\alpha$$

For semi-infinite crack, we have



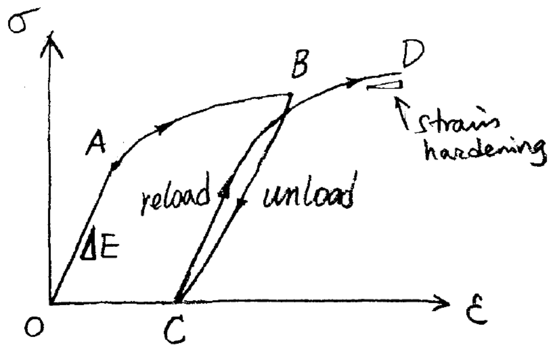
$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$
$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right), \qquad K_{II} = 3A\sqrt{\frac{\pi}{2}}$$
$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

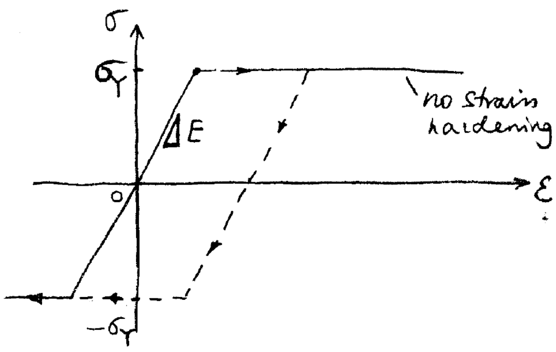
The constant K_I and K_{II} are called stress intensity factor

Lecture 11. Plasticity equations I

- Tensile test for a ductile material

For non-linear elastic material, the unloading curve is exactly the same as the loading one, even though the relation is not linear. If the unloading curve is different, then it is plastic





The real material can be complex, so we idealize **elastic-perfectly plastic (EPP)** materials

- Plasticity equations

Displacement, strain, stress fields and traction are the same.

Equilibrium condition

$$\sigma_{ij,i} + F_j = 0$$

Compatibility condition for the **total strain field**

$$\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl}$$

In general, both elastic and plastic strain fields are incompatible. Consider plastically deform the inclusion and squeeze it back into the elastic matrix. The elastic strain will adapt to the inserted inclusion and result in a final compatible total strain field

Constitutive equation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^{el}, \qquad \sigma_{ij} = \lambda\varepsilon_{kk}^{el}\delta_{ij} + 2\mu\varepsilon_{ij}^{el}, \qquad \lambda = 2\mu \cdot \frac{\nu}{1-2\nu}$$

There is a term 'elastic strain', but no such term as '~~elastic stress~~'

Hydrostatic and deviatoric components

Hydrostatic stress and strain (first **invariant**)

$$\bar{\sigma} = \frac{1}{3}\sigma_{kk}, \qquad \bar{\varepsilon} = \frac{1}{3}\varepsilon_{kk}, \qquad \bar{\sigma} = 3K\bar{\varepsilon}^{el}$$

Deviatoric stress and strain

$$s_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij}, \qquad e_{ij} = \varepsilon_{ij} - \bar{\varepsilon}\delta_{ij}, \qquad s_{ij} = 2\mu e_{ij}^{el}$$

We can obtain

$$\varepsilon_{ij}^{el} = \bar{\varepsilon}^{el}\delta_{ij} + e_{ij}^{el}, \qquad \sigma_{ij} = 3K\bar{\varepsilon}^{el}\delta_{ij} + 2\mu e_{ij}^{el}$$

Yield condition

Mathematically, the yield surface is described as

$$f(\sigma_{ij}) = f(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}) = 0$$

We already idealize the yield surface to be independent of stress rate, strain rate, etc. We also consider isotropic material, so coordinate transformation does not change the yield surface. Therefore, we need the **stress invariants**, with expressions in the principal axes

$$I_1 = \mathrm{tr}(\sigma_{ij}) = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = -\frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$I_3 = \det(\sigma_{ij}) = \sigma_1\sigma_2\sigma_3$$

Usually, the yield condition is independent of pressure by experiments. For deviatoric stress, the stress invariants become

$$J_1 = \mathrm{tr}(s_{ij}) = s_1 + s_2 + s_3 = 0$$

$$J_2 = \frac{1}{2}s_{ij}s_{ij} = -(s_1 s_2 + s_2 s_3 + s_3 s_1) = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2)$$

$$J_3 = \det(s_{ij}) = s_1 s_2 s_3 = \frac{1}{3}(s_1^3 + s_2^3 + s_3^3)$$

The yield surface of **isotropic material** becomes

$$f(J_2, J_3) = 0$$

Von Mises yield condition

One widely used yield condition is J_2-plasticity. The Von Mises criterion is

$$f(J_2) = J_2 - k^2 = 0$$

Tresca yield condition

The Tresca criterion is based on the maximum shear stress, and it has the form $f(J_2, J_3) = 0$. Using the Mohr circle, we write it in terms of the principal stress

$$\sigma_1 - \sigma_3 = 2k_T$$

Lecture 12. Plasticity equations II

- Yield condition & Yield stress

We usually report the yield stress σ_Y under **uniaxial tension** case. For Von Mises criterion

$$s_{ij} = \mathrm{diag}\left(\frac{2}{3}\sigma, -\frac{1}{3}\sigma, -\frac{1}{3}\sigma\right), \qquad J_2 = \frac{1}{3}\sigma_Y^2 = k^2, \qquad k = \frac{\sigma_Y}{\sqrt{3}}$$

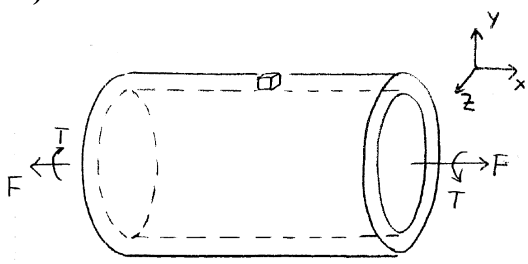
For Tresca criterion

$$\sigma_1 - \sigma_3 = \sigma_Y = 2k_T, \qquad k_T = \frac{\sigma_Y}{2}$$

Tension & shear (Taylor-Quinney experiment, 1931)

For Von Mises criterion

$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2, \qquad \tau_Y = \frac{\sigma_Y}{\sqrt{3}}$$



For Tresca criterion

$$\sigma_1 - \sigma_3 = \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2} = 2k_T, \qquad \tau_Y = \frac{\sigma_Y}{2}$$

Based on this experiment, Von Mises criterion seems to fit data better

- Flow rule in the plastic regime

For EPP material and Von Mises criterion, without material hardening, after yielding the stress state will always be on the yield surface

$$J_2 = \frac{1}{2}s_{ij}s_{ij} = k^2, \qquad \dot{J}_2 = s_{ij}\dot{s}_{ij} = 0$$

It turns out that the plastic strain is history dependent (related to the loading path) and is not a function of stress. We thus need the **incremental** theory, and the form is similar to the fluid mechanics. The **associative flow rule** states that plastic strain rate follows the direction of s_{ij}

$$\varepsilon_{ij}^{pl} = \int_0^t \dot{\varepsilon}_{ij}^{pl}(t)\,\mathrm{d}t, \qquad \dot{\varepsilon}_{ij}^{pl} = \frac{\tilde{\lambda}}{2\mu}s_{ij}, \qquad e_{ij}^{el} = \frac{1}{2\mu}s_{ij}$$

This flow rule implies that **the plastic strain (rate) has no volumetric part**

$$\dot{\varepsilon}_{kk}^{pl} = 0, \qquad \varepsilon_{kk}^{pl} = 0, \qquad \varepsilon_{ij}^{pl} = \bar{\varepsilon}^{pl}\delta_{ij} + e_{ij}^{pl}$$

The factor $\tilde{\lambda}$ is related to the rate of work done

$$\tilde{\lambda} = \frac{2\mu}{2k^2}\dot{W}, \qquad \dot{W} = s_{ij}\dot{e}_{ij}, \qquad \dot{W}_{\mathrm{total}} = \bar{\sigma}\dot{\bar{\varepsilon}} + \dot{W}$$

This can be shown as follows, using the yield surface $\dot{J}_2 = 0$ for EPP material

$$2\mu\dot{W} = s_{ij}\cdot 2\mu\left(\dot{e}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl}\right) = s_{ij}\dot{s}_{ij} + \tilde{\lambda}s_{ij}s_{ij} = 2\tilde{\lambda}k^2, \qquad \dot{W} = \frac{2\tilde{\lambda}}{2\mu}k^2$$

Lecture 13. Graphical representation of plasticity

- Yield surface in principal stress space

Von Mises criterion

Yield surface is a circular cylinder along the (1,1,1) diagonal direction

$$J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

In plane stress, $\sigma_3 = 0$ and the yield surface is an ellipse

$$J_2 = \frac{1}{3}(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2) = k^2, \qquad \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$

Tresca criterion

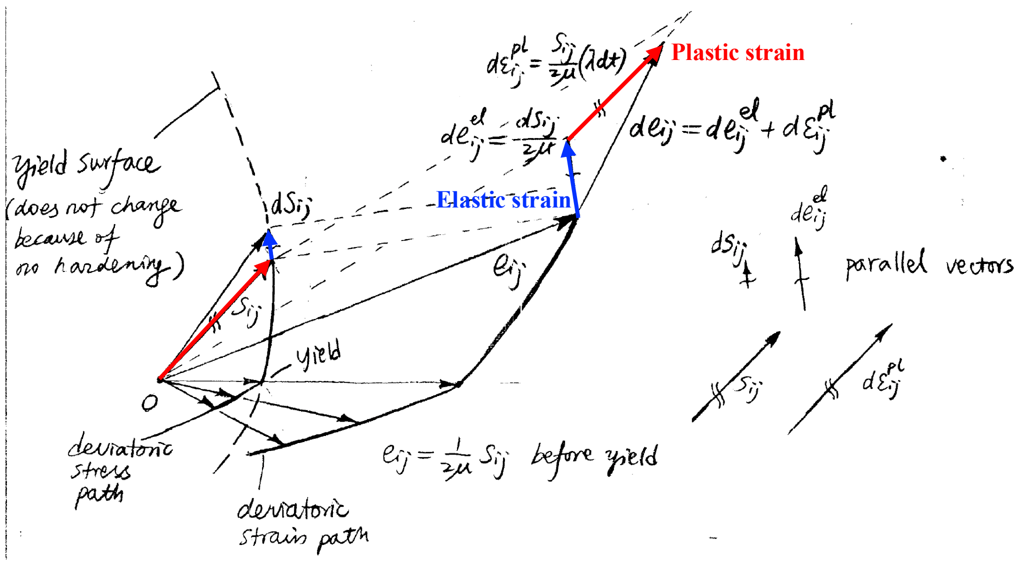
Yield surface is a hexagon inscribed in the Von Mises surface

$$|\sigma_1 - \sigma_2| = 2k_T \quad \text{or} \quad |\sigma_2 - \sigma_3| = 2k_T \quad \text{or} \quad |\sigma_3 - \sigma_1| = 2k_T$$

In plane stress, $\sigma_3 = 0$ and the yield surface is a hexagon inscribed in the Von Mises ellipse

$$|\sigma_1 - \sigma_2| = 2k_T \quad \text{or} \quad |\sigma_2| = 2k_T \quad \text{or} \quad |\sigma_1| = 2k_T$$

- Flow rule for general ductile materials

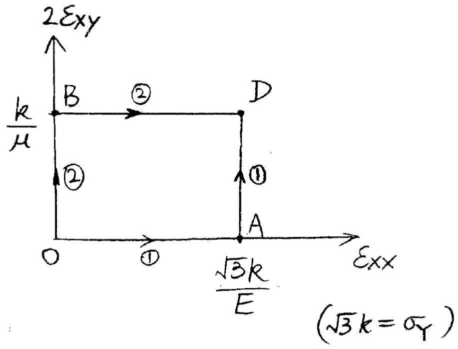


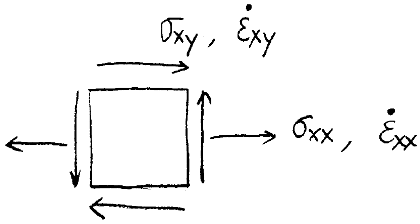
Elastic regime: s_{ij} and e_{ij} are proportional to each other

Plastic regime: s_{ij} follows the yield surface, e_{ij} is no longer proportional to s_{ij}

$$de_{ij}^{el} \parallel ds_{ij}, \qquad d\varepsilon_{ij}^{pl} \parallel s_{ij}$$

Lecture 14. Example plasticity problem: Tension & Shear





Assume the EPP material is incompressible

$$\nu = 0.5, \qquad E = 2\mu(1+\nu) = 3\mu$$

- Loading path 1 (Tension + Shear)

OA section: Elastic regime

$$\sigma_{xx} = E\varepsilon_{xx} = 3\mu\varepsilon_{xx}, \qquad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$$

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} = -\frac{1}{2}\varepsilon_{xx}$$

AD section: Plastic regime

Note that instantaneously after A, the plastic flow is in the tension direction. Plasticity strain rate satisfies the flow rule, and under incompressible assumption we have

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2}s_{ij}, \qquad \dot{W} = s_{ij}\dot{e}_{ij} = \sigma_{ij}\dot{\varepsilon}_{ij} = 2\sigma_{xy}\dot{\varepsilon}_{xy}$$

Therefore, we obtain the equation for shear strain and stress

$$\dot{\varepsilon}_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{W}}{2k^2}\sigma_{xy} = \frac{\dot{\sigma}_{xy}}{2\mu} + \frac{\dot{\varepsilon}_{xy}}{k^2}\sigma_{xy}^2, \qquad 2\mu\frac{\dot{\varepsilon}_{xy}}{k} = \frac{\frac{\dot{\sigma}_{xy}}{k}}{1-\left(\frac{\sigma_{xy}}{k}\right)^2}$$

Integration over time gives the solution

$$2\mu\frac{\varepsilon_{xy}(t)}{k} = \operatorname{arctanh}\left[\frac{\sigma_{xy}(t)}{k}\right], \qquad \frac{\sigma_{xy}(t)}{k} = \tanh\left[2\mu\frac{\varepsilon_{xy}(t)}{k}\right]$$

The Von Mises yield surface gives the normal stress history

$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2, \qquad \frac{\sigma_{xx}(t)}{k} = \sqrt{3}\cdot\sqrt{1-\left(\frac{\sigma_{xy}}{k}\right)^2} = \frac{\sqrt{3}}{\cosh\left[2\mu\frac{\varepsilon_{xy}(t)}{k}\right]}$$

The stress components at point A and D are

$$\sigma_{xx}(A) = \sqrt{3}k, \qquad \sigma_{xy}(A) = 0, \qquad \sigma_{xx}(D) = 1.12k, \qquad \sigma_{xy}(D) = 0.76k$$

- Loading path 2 (Shear + Tension)

OB section: Elastic regime

$$\sigma_{xx} = 2\mu\varepsilon_{xy}, \qquad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

BD section: Plastic regime

Note that instantaneously after B, the plastic flow is in the shear direction

$$\dot{\varepsilon}_{ij}^{pl} = \frac{\dot{W}}{2k^2} s_{ij}, \qquad \dot{W} = s_{ij}\dot{e}_{ij} = \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{xx}\dot{\varepsilon}_{xx}$$

Therefore, we obtain the equation for normal strain and stress

$$\dot{\varepsilon}_{xx} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{W}}{2k^2} s_{xx} = \frac{\dot{\sigma}_{xx}}{E} + \frac{\dot{\varepsilon}_{xx}}{3k^2}\sigma_{xx}^2, \qquad E\frac{\dot{\varepsilon}_{xx}}{\sqrt{3}k} = \frac{\frac{\dot{\sigma}_{xx}}{\sqrt{3}k}}{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2}$$

Integration over time gives the solution

$$E\frac{\varepsilon_{xx}(t)}{\sqrt{3}k} = \operatorname{arctanh}\left[\frac{\sigma_{xx}(t)}{\sqrt{3}k}\right], \qquad \frac{\sigma_{xx}(t)}{\sqrt{3}k} = \tanh\left[E\frac{\varepsilon_{xx}(t)}{\sqrt{3}k}\right]$$

The Von Mises yield surface gives the shear stress history

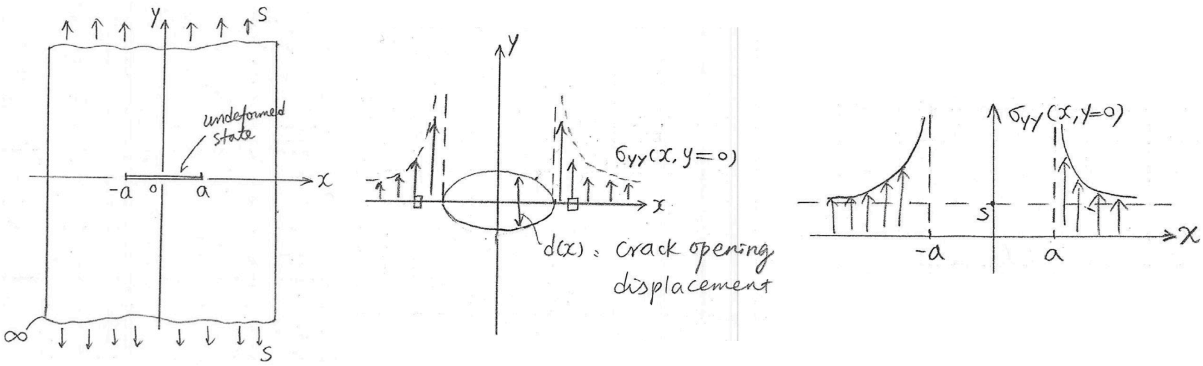
$$J_2 = \frac{1}{3}\sigma_{xx}^2 + \sigma_{xy}^2 = k^2, \qquad \frac{\sigma_{xy}(t)}{k} = \sqrt{1 - \left(\frac{\sigma_{xx}}{\sqrt{3}k}\right)^2} = \frac{1}{\cosh\left[\sqrt{3}\mu\frac{\varepsilon_{xx}(t)}{k}\right]}$$

The stress components at point B and D are

$$\sigma_{xx}(B) = 0, \qquad \sigma_{xy}(B) = k, \qquad \sigma_{xx}(D) = 1.31k, \qquad \sigma_{xy}(D) = 0.64k$$

Lecture 15. Linear elastic fracture mechanics (LEFM)

- Slit-like cracks



The undeformed crack has a length of $2a$. The applied tensile loading is $\sigma_{yy}^A = S$. The crack opens with displacement $d(x)$ and stress singularity appears at $x = \pm a$. For $x > a$ we have

$$\sigma_{yy}(x, y=0) \sim \frac{K_I}{\sqrt{2\pi r}}, \qquad r = x - a$$

This can be solved as a contact problem with equivalent loading $-p_y(x)$ outside the crack

$$\tilde{u}_y(x) = \frac{\kappa+1}{4\pi\mu}\int_{-\infty}^{+\infty} p_y(x')\ln|x-x'|\,\mathrm{d}x'\,, \qquad \frac{\mathrm{d}\tilde{u}_y(x)}{\mathrm{d}x} = \frac{\kappa+1}{4\pi\mu}\int_{-\infty}^{+\infty}\frac{p_y(x')}{x-x'}\,\mathrm{d}x'$$

The integral is in fact only over $|x| > a$. Within the same domain we have

$$g(x) = \frac{4\pi\mu}{\kappa+1}\cdot\frac{\mathrm{d}\tilde{u}_y(x)}{\mathrm{d}x} = 0, \qquad p_y(x) = \frac{A + B/x}{\sqrt{1-(a/x)^2}}$$

By symmetry $p_y(x) = p_y(-x)$ we have $B = 0$. The infinity condition gives $A = -S$. Then the normal stress on the crack plane is

$$\sigma_{yy}(x, y=0) = -p_y(x) = \frac{S|x|}{\sqrt{x^2-a^2}}, \qquad |x| > a$$

Stress intensity factor

In the polar coordinate, with the origin at $x = a$ we have

$$\sigma_{\theta\theta}(r, \theta=0) = \frac{S(a+r)}{\sqrt{(a+r)^2-a^2}} \sim S\sqrt{\frac{a}{2}}\cdot\frac{1}{\sqrt{r}} = \frac{K_I}{\sqrt{2\pi r}}, \qquad \text{as } r \to 0$$

The **stress intensity factor** for **Mode I opening** crack is

$$K_I = S\sqrt{\pi a}, \qquad \text{unit is } [\mathrm{Pa}\cdot\mathrm{m}^{1/2}]$$

It is proportional to the applied loading S and the square root of crack half-length $\sqrt{a}$.

Crack opening displacement

From the contact problem, we can obtain

$$\tilde{u}_y(x) = -\frac{\kappa+1}{4\pi\mu} Sa\sqrt{1-(x/a)^2}, \qquad |x| < a, \qquad y = 0^-$$

The crack opening displacement, under plane strain assumption, is shown as

$$d(x) = -2\tilde{u}_y(x), \qquad d(x) = \frac{2(1-\nu)}{\mu} Sa\sqrt{1-(x/a)^2}$$

The maximum is proportional to the applied loading S and the crack half-length a

- Enthalpy of the crack

A mechanical system **under external load** will **evolve in the direction to reduce enthalpy**.

Enthalpy is the energy minus the work done by the loading mechanism

$$H = E - \Delta W_{\mathrm{LM}}$$

As an example, for a spring with external forcing F, its enthalpy is

$$H = \frac{1}{2}kx^2 - Fx$$

Under the loading mechanism, at equilibrium state the enthalpy is minimized

$$\frac{\partial H}{\partial x} = 0, \qquad x = \frac{F}{k}$$

For a linear elastic medium with volume Ω under traction T_i on section S_t, the enthalpy is

$$H = E - \Delta W_{\mathrm{LM}} = \int_{\Omega} \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V - \int_{S_t} T_j u_j\,\mathrm{d}S$$

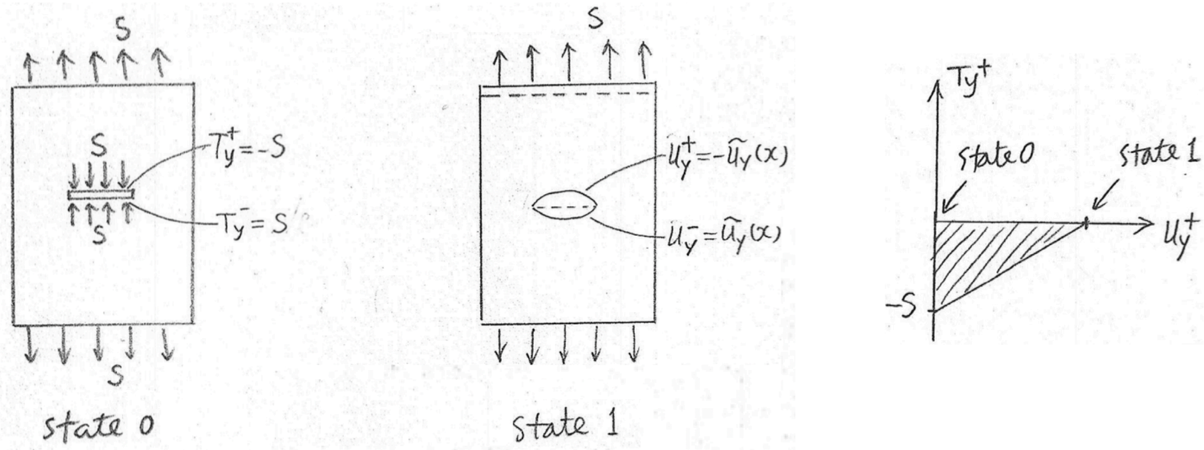
For a body with no pre-existing internal stress, then we have $H = -E$ at equilibrium state

When the system contains no crack, the energy and enthalpy are denoted as E_0 and H_0. Then the crack with length a appears, and they change to E_1 and H_1 under plane strain condition. Now we can calculate the energy and enthalpy changes ΔE and ΔH

$$E_0 = \frac{1}{2}\sigma_{ij}^A\varepsilon_{ij}^A V, \qquad \Delta E = \frac{1-\nu}{2\mu}S^2\pi a^2, \qquad \Delta H = -\Delta E = -\frac{1-\nu}{2\mu}S^2\pi a^2$$

Lecture 16. Griffith criterion

➢ Derivation of enthalpy of the crack



When there is no applied load $\sigma_{yy}^A = S$, the energy is 0. At initial state 0, we have

$$E_0 = \frac{1}{2}\sigma_{yy}^A \varepsilon_{yy}^A V, \qquad H_0 = -E_0 = -\frac{1}{2}\sigma_{yy}^A \varepsilon_{yy}^A V$$

Since enthalpy is a state variable, we consider a reversible way to go between state 0 and state 1. We gradually reduce the traction force T_y^+ and T_y^- to zero, and the two surfaces of the crack will reach the crack opening displacement

The work done to the system, which is negative as the traction and displacement directions are opposite, equals the enthalpy change

$$\Delta H = 2\Delta W^+ = -\frac{S}{2}\int_{-a}^{a} d(x)\,\mathrm{d}x = -\frac{1-\nu}{2\mu}S^2\pi a^2$$

The enthalpy is lowered by having the crack in the solid, because it allows the applied stress to do more work

➢ Griffith criterion (1921)

The (elastic) driving force for the crack extension is defined as

$$f_{\mathrm{el}} \equiv -\frac{\partial(\Delta H)}{\partial(2a)} = \frac{1-\nu}{2\mu}S^2\pi a, \qquad f_{\mathrm{el}} = \frac{1-\nu}{2\mu}K_I^2$$

It seems that all cracks are unstable as soon as a stress is applied. However, creating a new crack or growing an existing crack requires the creation of new surfaces, which costs extra energy related to the surface energy (per unit area) of the solid, denoted as γ_s

The change of Gibbs free energy of the system (per unit length), including both the (elastic) enthalpy and the surface energy, is written as

$$\Delta G = \Delta H + \gamma_s \cdot 2 \cdot 2a = \Delta H + 4\gamma_s a$$

The total driving force now becomes

$$f_{\text{tot}} \equiv -\frac{\partial(\Delta G)}{\partial(2a)} = \frac{1-\nu}{2\mu} S^2 \pi a - 2\gamma_s$$

The critical crack size is

$$2a_c = \frac{8\mu}{\pi(1-\nu)} \cdot \frac{\gamma_s}{S^2}$$

Under a given applied stress S, cracks with length $2a < 2a_c$ are stable (i.e., do not grow), and cracks with length $2a > 2a_c$ are unstable (i.e., lead to fracture)

Similarly, for a given crack size $2a$, the critical stress is

$$S_c = \sqrt{\frac{8\mu\gamma_s}{\pi(1-\nu)(2a)}}$$

In general, the critical condition is $f_{\text{el}} = 2\gamma_s$

Using the Kolosov constant, for Mode I crack opening the results are summarized as

$$d(x) = \frac{\kappa+1}{2\mu} Sa\sqrt{1-(x/a)^2}$$

$$\Delta H = -\frac{\kappa+1}{8\mu} S^2 \pi a^2$$

$$f_{\text{tot}} = \frac{\kappa+1}{8\mu} K_I^2 - 2\gamma_s$$

$$S_c = \sqrt{\frac{16\mu\gamma_S}{\pi(\kappa+1)a}}$$

Lecture 17. Linear Elastic Fracture Mechanics (LEFM)

- Energy release rate

The Griffith criterion can be written in terms of the relation between energy release rate $\mathcal{G}$ and the critical energy release rate $\mathcal{G}_c$. For plane strain Mode I loading, we have

$$\mathcal{G} = f_{\text{el}} = \frac{\pi(1-\nu)}{2\mu}\left(\sigma_{yy}^{A}\right)^2 a, \qquad \mathcal{G}_c = 2\gamma_S$$

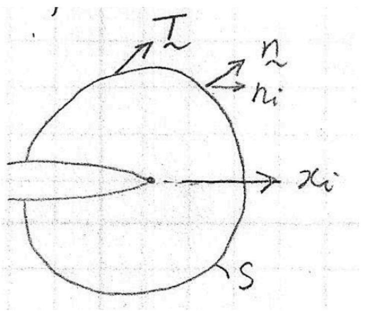
We can also write it as

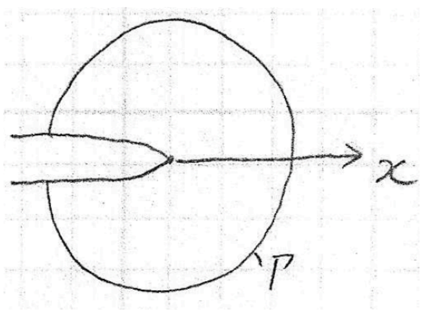
$$\mathcal{G} = \frac{\pi}{E'}\left(\sigma_{yy}^{A}\right)^2 a = \frac{K_I^2}{E'}, \qquad K_I = \sigma_{yy}^{A}\sqrt{\pi a}, \qquad E' = \frac{E}{1-\nu^2}$$

Under plane strain assumption, for general loading the energy release rate is

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}, \qquad \mathcal{G} \geq \mathcal{G}_c$$

- J-integral (Eshelby 1951, Rice 1968)





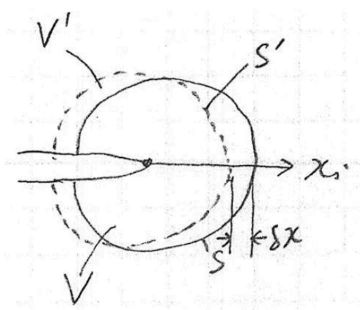
The force on an elastic singularity in the x_i-direction is

$$J_i = \int_S \left(w n_i - T_j u_{j,i}\right) \mathrm{d}S$$

In the expression, w is the elastic energy density, and the first term is a weighted sum of w on the surface. In 2D problem, we have

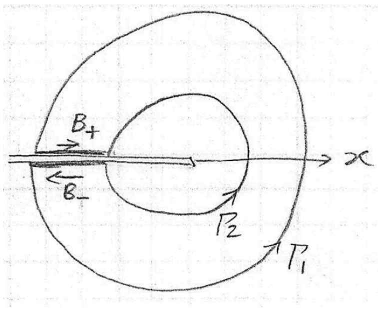
$$J = J_x = \int_\Gamma w\mathrm{d}y - T_j \frac{\partial u_j}{\partial x} \mathrm{d}s$$

The propagation of the crack tip can be considered as shifting the contour in the opposite direction. Although the strain and stress fields are not exactly the same for the two scenarios,



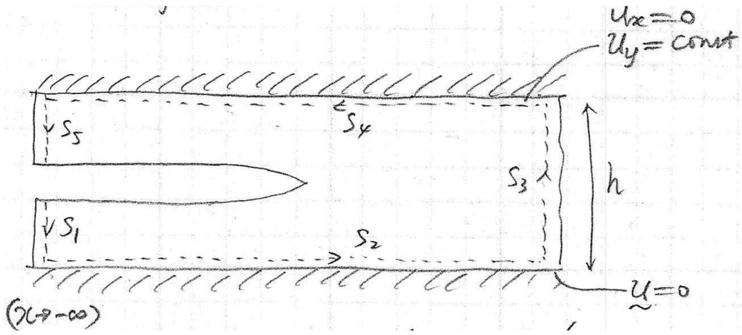
but the enthalpy change and thus the J-integral are the same

Example 1: J-integral is independent of contours, as long as the singularity is enclosed



Define the new contour as $\Gamma = \Gamma_1 + B_+ - \Gamma_2 + B_-$. The integral on Γ is zero as no singularity is included. On the lines B_+ and B_-, the traction is zero and $\mathrm{d}y = 0$. Therefore, the contour integrals on Γ_1 and Γ_2 are equal

Example 2: J-integral considered in Rice (1968)



We can evaluate each segment separately

$$J(S_2) = J(S_4) = \int_\Gamma w\mathrm{d}y - \boldsymbol{T} \cdot \frac{\partial \boldsymbol{u}}{\partial x} \mathrm{d}s = 0$$

$$J(S_1) = J(S_5) = \int_\Gamma w\mathrm{d}y - \boldsymbol{T} \cdot \frac{\partial \boldsymbol{u}}{\partial x} \mathrm{d}s = 0$$

The only non-zero contribution comes from S_3

$$J = J(S_3) = \int_\Gamma w\mathrm{d}y - \boldsymbol{T} \cdot \frac{\partial \boldsymbol{u}}{\partial x} \mathrm{d}s = wh$$

The crack propagation converts one unit of strained material ahead of the crack tip, to one unit of unstrained material behind the crack tip

Lecture 18. Linear Elastic Fracture Mechanics (LEFM)

➢ J-integral (Eshelby 1951, Rice 1968)

Example 3: J-integral for Mode I crack

The singular solution around the crack tip is given as

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right)$$
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$

We can show that

$$J = \int_\Gamma w\mathrm{d}y - \boldsymbol{T}\cdot\frac{\partial \boldsymbol{u}}{\partial x}\mathrm{d}s = \frac{1-\nu}{2\mu}K_I^2 = \frac{K_I^2}{E'}$$

Example 4: J-integral for a blunted ductile crack tip (no stress singularity)

The contour is chosen to be very close to the blunted crack tip where $\boldsymbol{T} = \boldsymbol{0}$

$$J = \int_\Gamma w\mathrm{d}y - \boldsymbol{T}\cdot\frac{\partial \boldsymbol{u}}{\partial x}\mathrm{d}s = \int_\Gamma w\mathrm{d}y$$

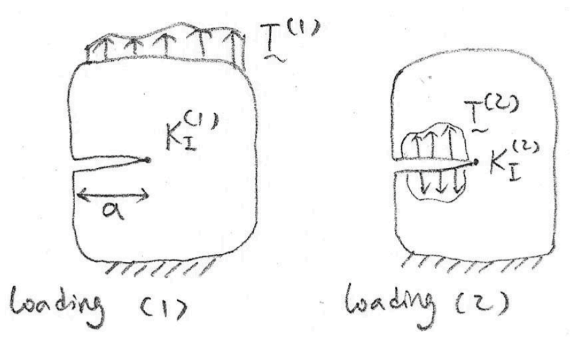
At the blunted crack tip, the stress state is uniaxial

➢ Fracture criterion & Applicability of LEFM

$$\mathcal{G} \geq \mathcal{G}_c \quad \Leftrightarrow \quad K_I \geq K_{IC}$$

LEFM is applicable when the plastic zone is very limited (inside K-field). The comparison between the sizes of K-dominated zone and plastic zone is important

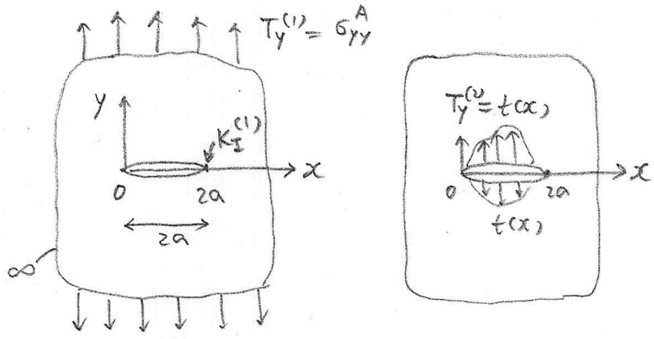
➢ Stress intensity factor under arbitrary loading (**Reciprocity theorem**; Rice, 1972)



Under loading 2 with the sample/crack geometry unchanged, the stress intensity factor is

$$K_I^{(2)} = \frac{E'}{2K_I^{(1)}}\int_\Gamma T_i^{(2)}\frac{\partial u_i^{(1)}}{\partial a}\,\mathrm{d}\Gamma$$

Example: Slit-like crack



For slit-like crack, the displacement on the crack surface is

$$u_y^{(1)} = \pm\frac{2\sigma_{yy}^A}{E'}\sqrt{x(2a-x)}, \qquad \frac{\partial u_y^{(1)}}{\partial(2a)} = \pm\frac{\sigma_{yy}^A}{E'}\sqrt{\frac{x}{2a-x}}$$

We can obtain the stress intensity factor under the new loading scenario as

$$K_I^{(2)} = \frac{E'}{2\sigma_{yy}^A\sqrt{\pi a}}\int_0^{2a} 2t(x)\frac{\sigma_{yy}^A}{E'}\sqrt{\frac{x}{2a-x}}\,\mathrm{d}x = \frac{1}{\sqrt{\pi a}}\int_0^{2a} t(x)\sqrt{\frac{x}{2a-x}}\,\mathrm{d}x$$

If the crack is self-consistent, for a uniform normal loading $t(x) = S$ we have the same result

$$K_I^{(2)} = \frac{S}{\sqrt{\pi a}}\int_0^{2a}\sqrt{\frac{x}{2a-x}}\,\mathrm{d}x = S\sqrt{\pi a} = K_I^{(1)}$$

This can be interpreted from the superposition of state 0 and state 1

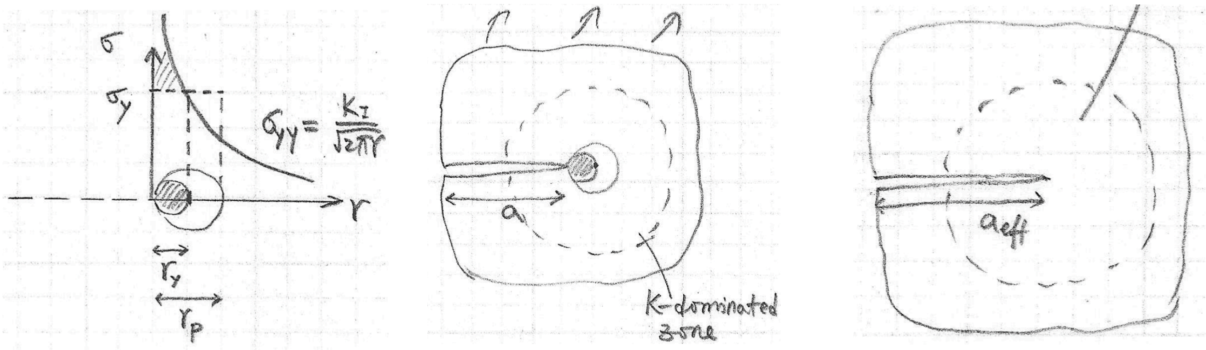
If the normal loading is a dipole force $t(x) = F\delta(x-a)$

$$K_I^{(2)} = \frac{1}{\sqrt{\pi a}}\int_0^{2a} F\delta(x-a)\sqrt{\frac{x}{2a-x}}\,\mathrm{d}x = \frac{F}{\sqrt{\pi a}}$$

In this case, the crack is stable as K_I decreases with the crack length

Lecture 19. Elastic-Plastic Fracture Mechanics (EPFM)

➢ Size of the plastic zone (Irwin's approach)



At the yielding stress σ_Y we have

$$\frac{K_I}{\sqrt{2\pi r}} = \sigma_Y, \qquad r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2, \qquad r_p = 2r_y = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$$

Now the loading in $r < r_y$ is reduced and must be transferred to other region. The estimated plastic region is extended to $r_p = 2r_y$. This plastic yielding can increase the toughness K_{Ic}, and also changes K_I. A rough estimate is to consider the effective crack length

$$K_I^{\text{eff}} = \frac{P}{B\sqrt{W}} f\left(\frac{a_{\text{eff}}}{W}\right), \qquad K_I^{\text{eff}} = \sigma_{yy}^A \sqrt{\pi a_{\text{eff}}}, \qquad a_{\text{eff}} = a + r_y$$

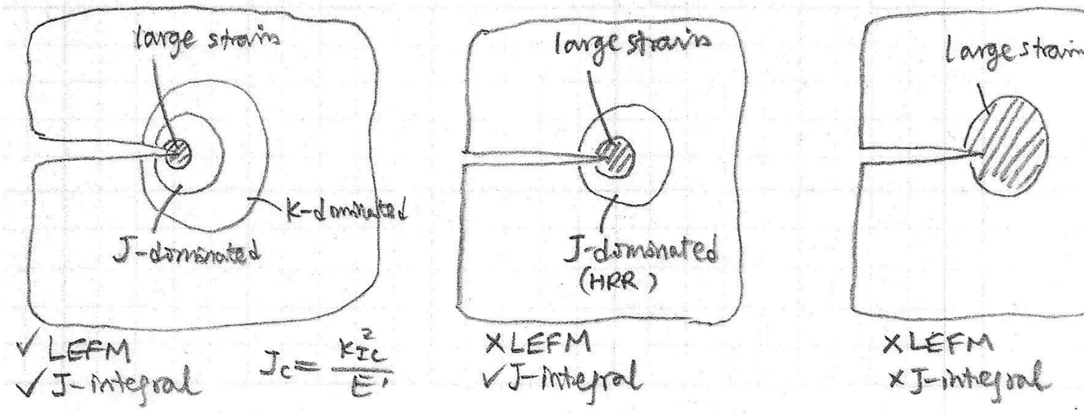
➢ HRR solution (Hutchinson 1968, Rice & Rosengren 1968)

We can approximate a plastic material as a nonlinear elastic material

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$

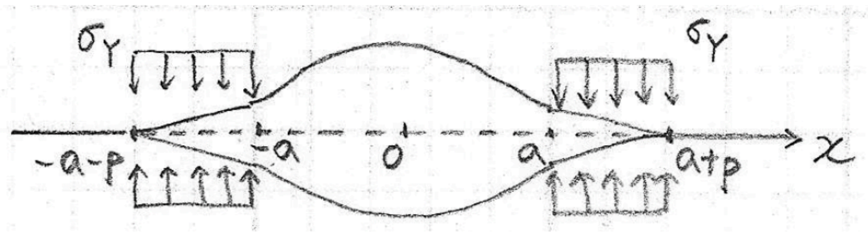
The crack tip is completely characterized by the J-integral, and the stress and strain fields are given by the HRR singularity

$$\sigma_{ij} = k_1\left(\frac{J}{r}\right)^{\frac{1}{n+1}}, \qquad \varepsilon_{ij} = k_2\left(\frac{J}{r}\right)^{\frac{n}{n+1}}, \qquad \sigma \cdot \varepsilon \propto \frac{1}{r}$$



As the loading increases, the K-dominated zone is the same, while both the J-dominated and plastic zones expand. The applicability of LEFM and J-integral is illustrated above

- Strip-yield model



Under yielding, the crack tip at a is blunted. Now we cut the region ahead of the crack tip to $a+\rho$ and close it partially with the yield stress σ_Y

At $x = a+\rho$ the stress field should be non-singular, and we have $K_I^{\text{tot}} = 0$. The closure force contributes to the stress intensity factor as

$$K_{\text{closure}} = \frac{1}{\sqrt{\pi a'}} \int_{-a'}^{a'} t(x) \sqrt{\frac{a'+x}{a'-x}}\, \mathrm{d}x$$

$$= -\frac{\sigma_Y}{\sqrt{\pi(a+\rho)}} \int_a^{a+\rho} \left(\sqrt{\frac{a+\rho+x}{a+\rho-x}} + \sqrt{\frac{a+\rho-x}{a+\rho+x}} \right) \mathrm{d}x$$

$$= -2\sigma_Y \sqrt{\frac{a+\rho}{\pi}} \cdot \arccos\left(\frac{a}{a+\rho}\right)$$

Therefore, the non-singular condition gives

$$\sigma_{yy}^A \sqrt{\pi(a+\rho)} = 2\sigma_Y \sqrt{\frac{a+\rho}{\pi}} \cdot \arccos\left(\frac{a}{a+\rho}\right), \qquad \frac{a}{a+\rho} = \cos\left(\frac{\pi}{2}\frac{\sigma_{yy}^A}{\sigma_Y}\right)$$

Approximately, we have

$$\rho \approx \frac{\pi^2}{8}\left(\frac{\sigma_{yy}^A}{\sigma_Y}\right)^2 a = \frac{\pi}{8}\left(\frac{K_I}{\sigma_Y}\right)^2 \approx 0.393\left(\frac{K_I}{\sigma_Y}\right)^2, \qquad K_I = \sigma_{yy}^A \sqrt{\pi a}$$

As a comparison, Irwin's approach gives a similar estimation of the plastic zone

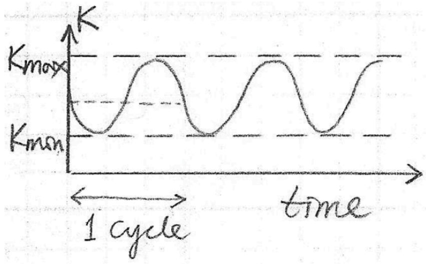
$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 \approx 0.318\left(\frac{K_I}{\sigma_Y}\right)^2$$

Lecture 20. Fatigue

Under cyclic loading, crack can grow and structure can fracture even when $K_I^{max} < K_{Ic}$

- Paris law

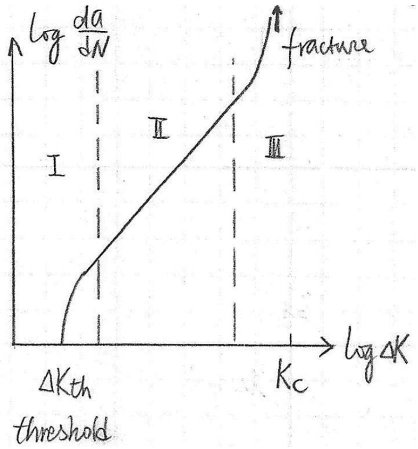
Cyclic loading leads to a cyclic K. The amplitude and ratio are defined as



$$\Delta K = K_{\max} - K_{\min}, \qquad R = \frac{K_{\min}}{K_{\max}}$$

For example, zero-mean sinusoidal loading gives $R = -1$

When the plastic zone is fully enclosed within the K-field, the crack growth rate (**per cycle**) is found to be



$$\frac{\mathrm{d}a}{\mathrm{d}N} = f(\Delta K, R)$$

Paris law states the power law expression in region II

$$\frac{\mathrm{d}a}{\mathrm{d}N} \cong C(\Delta K)^m, \qquad 2 \le m \le 4$$

This relation works for crack growth rate in the range of 10^{-9} to 10^{-6} m/cycle in experiments

A more general expression in all three regions is

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C(\Delta K)^m \frac{(1 - \Delta K_{\text{th}}/\Delta K)^p}{(1 - K_{\max}/K_c)^q}$$

where $C, m, p, q, \Delta K_{\text{th}}, K_c$ are material constants

- Slit-like crack in an infinite plate

Consider the following setup

$$K = \sigma_{yy}^A \sqrt{\pi a}, \qquad K_{\max} = S\sqrt{\pi a} = \Delta K, \qquad K_{\min} = 0, \qquad R = 0$$

We want to find the number of cycles until fracture, i.e. before the crack size reaches a_c

$$K = S\sqrt{\pi a_c} = K_{Ic}, \qquad a_c = \frac{1}{\pi}\left(\frac{K_{Ic}}{S}\right)^2$$

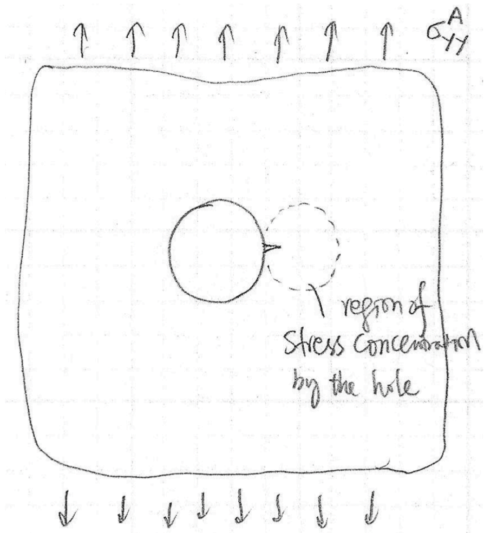
Assume the power law expression, we have

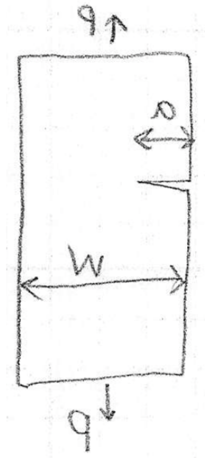
$$\frac{\mathrm{d}a}{\mathrm{d}N} = C(\Delta K)^m = C \cdot S^m (\pi a)^{m/2}, \qquad \frac{\mathrm{d}N}{\mathrm{d}a} = \frac{1}{C S^m \pi^{m/2}} \cdot a^{-m/2}$$

When $m > 2$ we have the following result

$$N_f = \frac{1}{\left(\frac{m}{2} - 1\right) C S^m \pi^{m/2}} \cdot \left(a_0^{-\frac{m}{2}+1} - a_c^{-\frac{m}{2}+1}\right)$$

➢ Crack on the surface of a hole





The circular hole leads to a stress concentration factor of 3. The stress intensity factor can be obtained in the limit of $W \to \infty$ as a

$$K_I = 1.122\,\sigma\sqrt{\pi a} = 1.122 \times 3S\sqrt{\pi a}$$

Then the maximum loading cycle becomes

$$N_f = \frac{1}{\left(\frac{m}{2} - 1\right) C \cdot (3.366S)^m \pi^{m/2}} \cdot \left(a_0^{-\frac{m}{2}+1} - a_c^{-\frac{m}{2}+1} \right)$$