Banana-Doughnut Kernels

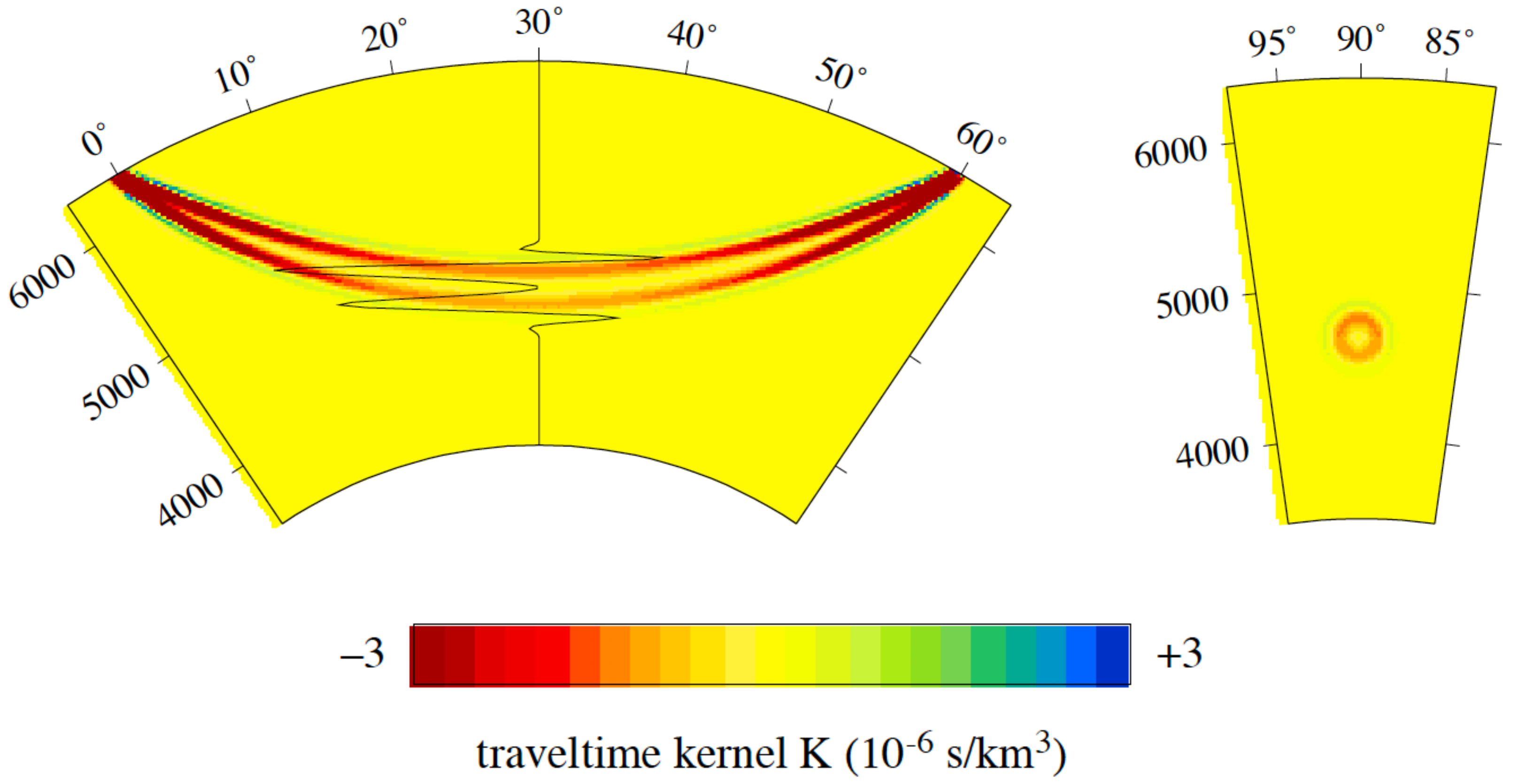
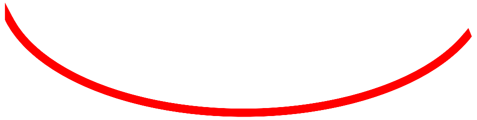
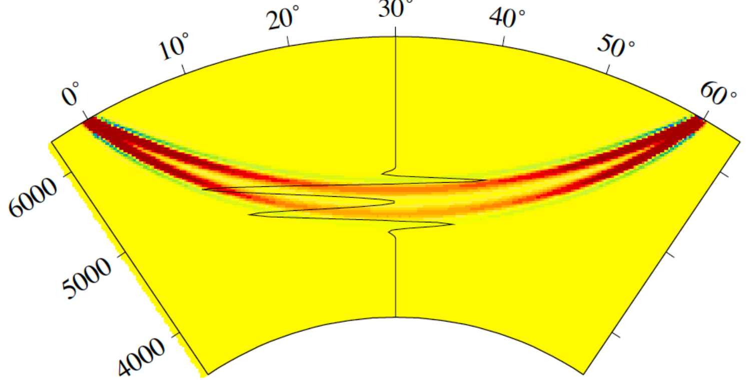


Figure from:
Hung, S.-H., Dahlen, F. A., & Nolet, G. **(2000)**. **Fréchet kernels** for **finite-frequency** traveltimes—II. Examples. *Geophysical Journal International*, *141*(1), 175–203.

What is a Kernel?





$$\delta T = \int_{\oplus} K_v(\boldsymbol{r}) \cdot \frac{\delta v(\boldsymbol{r})}{v(\boldsymbol{r})}\, d^3\boldsymbol{r}$$

Ray Theory

Eikonal equation

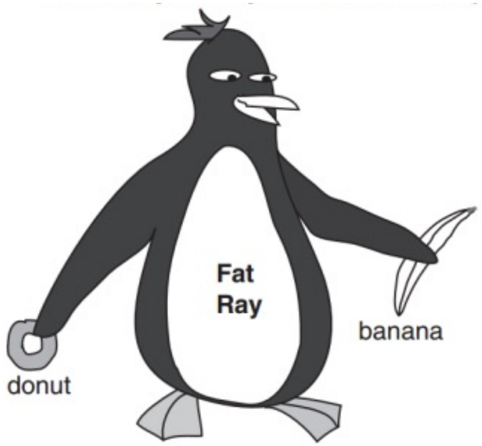
$$\omega \to \infty$$

$$\oplus \to \text{Raypath}$$

Finite-frequency Theory

Elastic wave equation

$$\rho\ddot{\mathbf{u}} - \nabla \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}\mathbf{u})] = \boldsymbol{f}$$



How to compute the banana-doughnut kernels?

Tromp, J., Tape, C., & Liu, Q. (2005). Seismic tomography, adjoint methods, time reversal and banana-doughnut kernels. *Geophysical Journal International*.

Zhao, L., & Chevrot, S. (2011). An efficient and flexible approach to the calculation of three-dimensional full-wave Fréchet kernels for seismic tomography-I. Theory. *Geophysical Journal International*.

Plessix, R.-E. (2006). A review of the adjoint-state method for computing the gradient of a functional with geophysical applications. *Geophysical Journal International*.

To compute the kernel, our goal is:

$$\delta T = \int_{\oplus} K_v(\boldsymbol{r}) \cdot \delta v(\boldsymbol{r})\, d^3\boldsymbol{r}$$

$$K_v(\boldsymbol{r}) = \frac{\delta T}{\delta v}(\boldsymbol{r})$$

Construct the linear relationship between model and data perturbations!

Where does our observation come from? …… Waveform $\boldsymbol{u}(\boldsymbol{r}, t)$!

$$\delta T_n(\boldsymbol{r}_R) = \int_0^T J^{T_n}(\boldsymbol{r}_R, t) \cdot \delta u_n(\boldsymbol{r}_R, t)\, dt$$

Linear Functional

between data and waveform

At receiver $\boldsymbol{r}_R$

n-th waveform component

Seismogram ends at T

Fundamental Goal: Linear relationship between model and waveform perturbations

$$\delta \boldsymbol{m}(\boldsymbol{r}_Q) \xrightarrow{?} \delta \boldsymbol{u}(\boldsymbol{r}_R, t)$$

At specific point $\boldsymbol{r}_Q$ in the Earth

In reference model:

$$\rho\ddot{\boldsymbol{u}} - \nabla\cdot\underbrace{[\boldsymbol{C}:(\boldsymbol{\nabla u})]}_{\boldsymbol{\sigma}} = \boldsymbol{f}$$

In perturbed model:

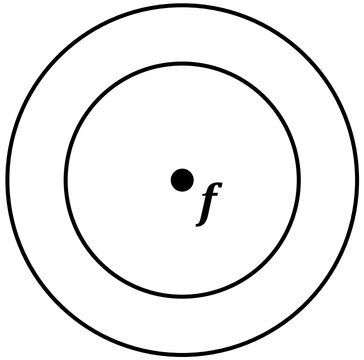
$$(\rho+\delta\rho)(\ddot{\boldsymbol{u}}+\delta\ddot{\boldsymbol{u}}) - \boldsymbol{\nabla}\cdot\{(\boldsymbol{C}+\delta\boldsymbol{C}):[\boldsymbol{\nabla}(\boldsymbol{u}+\delta\boldsymbol{u})]\} = \boldsymbol{f}$$

Subtraction:

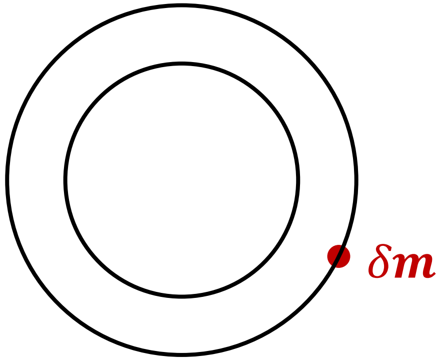
(With first-order Born)

$$\rho\delta\ddot{\boldsymbol{u}} - \nabla\cdot[\boldsymbol{C}:(\nabla\delta\boldsymbol{u})] = \boxed{\cancel{-\delta\rho\ddot{\boldsymbol{u}}} + \nabla\cdot[\delta\boldsymbol{C}:(\nabla\boldsymbol{u})]}$$

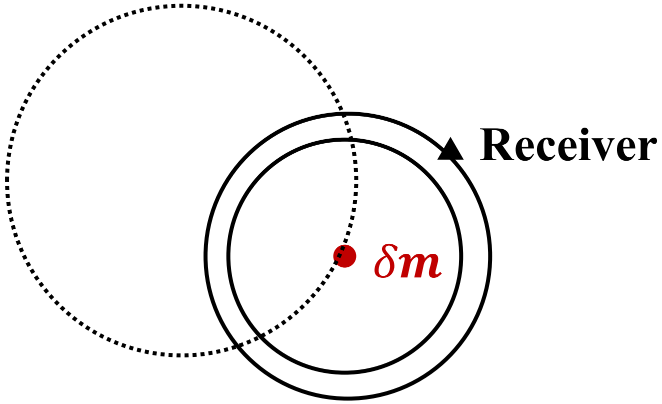
δm serves as scatters



$\boldsymbol{u}(\boldsymbol{r}, t)$



Scatter with $\delta\boldsymbol{f}(\boldsymbol{r}_Q, t)$



$\delta\boldsymbol{u}(\boldsymbol{r}, t)$

Background wavefield: $\rho\ddot{\boldsymbol{u}} - \nabla\cdot[\boldsymbol{C}:(\boldsymbol{\nabla u})] = \boldsymbol{f}$

Wavefield perturbation: $\rho\delta\ddot{\boldsymbol{u}} - \nabla\cdot[\boldsymbol{C}:(\nabla\delta\boldsymbol{u})] = \nabla\cdot\underbrace{[\delta\boldsymbol{C}:(\nabla\boldsymbol{u})]}_{\delta\boldsymbol{\sigma}}$

Using representation theorem and the Green tensor:

$$\delta\boldsymbol{u}(\boldsymbol{r}_R,t) = \int_{\oplus}\int_0^t \boldsymbol{G}(\boldsymbol{r}_R,t-\tau;\boldsymbol{r}_Q)\cdot[\boldsymbol{\nabla}^{\mathrm{Q}}\cdot\delta\boldsymbol{\sigma}(\boldsymbol{r}_Q,\tau)]\,d\tau\,d^3\boldsymbol{r}_Q$$

Scattered wavefield

Background wavefield

Using Gauss theorem, free surface condition and reciprocity:

$$\delta\boldsymbol{u}(\boldsymbol{r}_R,t) = \int_{\oplus}\int_0^t [\boldsymbol{\nabla}^{\mathrm{Q}}\boldsymbol{G}(\boldsymbol{r}_Q,t-\tau;\boldsymbol{r}_R)]^{213}:\delta\boldsymbol{\sigma}(\boldsymbol{r}_Q,\tau)\,d\tau\,d^3\boldsymbol{r}_Q$$

Backward wavefield

Forward wavefield

Waveform Kernel

$$K^{u_n}_{C_{ijkl}}(\boldsymbol{r}_Q,t) = \frac{\delta u_n}{\delta C_{ijkl}}(\boldsymbol{r}_Q,t) = -\varepsilon^{\dagger}_{ij}(\boldsymbol{r}_Q,t) * \varepsilon_{kl}(\boldsymbol{r}_Q,t)$$

Backward wavefield
(Adjoint wavefield)

Forward wavefield

Corresponding adjoint source: A point source at receiver $\boldsymbol{r}_R$ in n direction

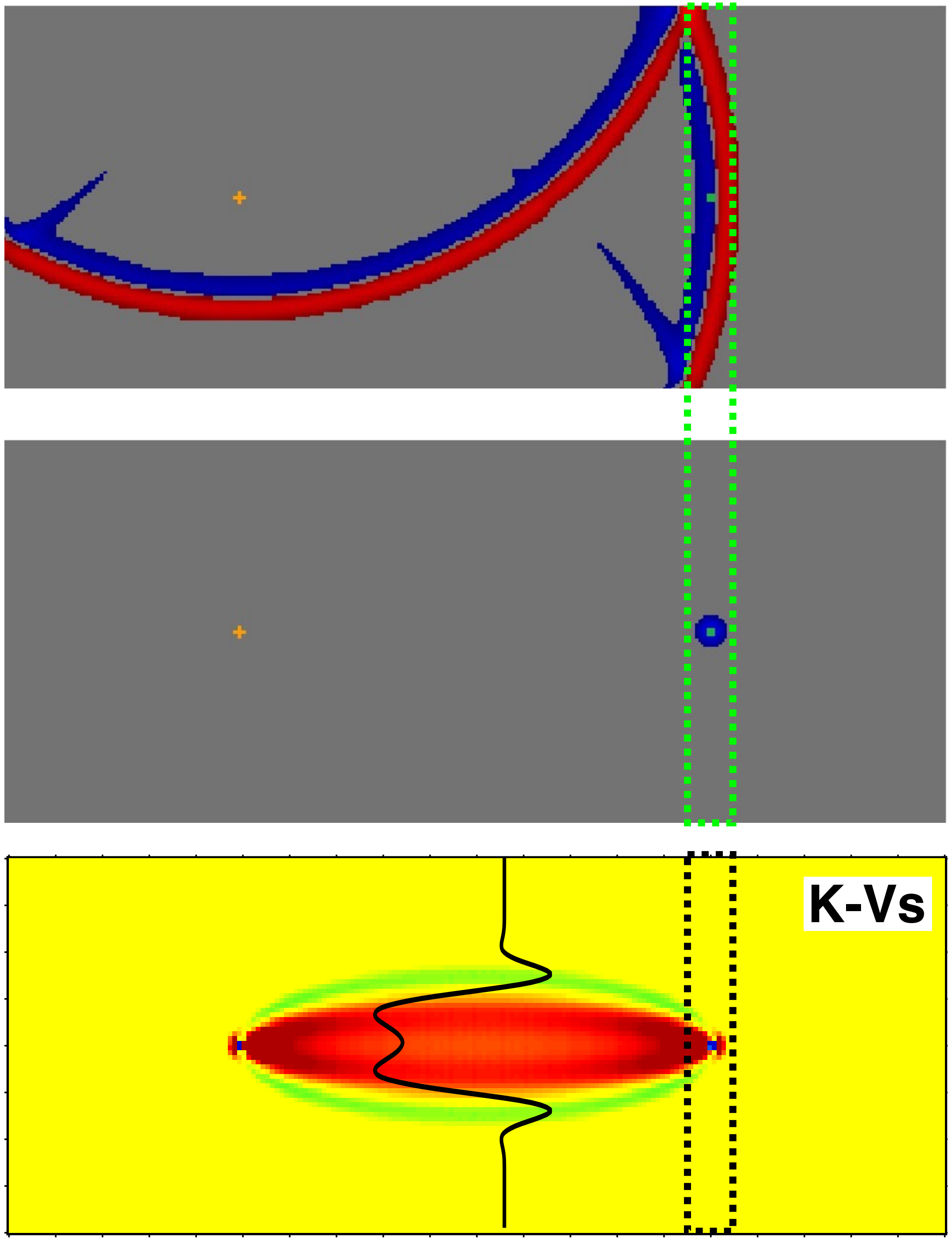
From Waveform Kernel to Travel Time Kernel

$$\delta T_n(\boldsymbol{r}_R) = \int_0^T J^{T_n}(\boldsymbol{r}_R,t)\cdot\delta u_n(\boldsymbol{r}_R,t)\,dt \longrightarrow K^{T_n}_{C_{ijkl}}(\boldsymbol{r}_Q) = \int_0^T J^{T_n}(\boldsymbol{r}_R,t)\cdot K^{u_n}_{C_{ijkl}}(\boldsymbol{r}_Q,t)dt$$

$$\boxed{K^{T_n}_{C_{ijkl}}(\boldsymbol{r}_Q) = -\int_0^T \varepsilon^{\dagger}_{ij}(\boldsymbol{r}_Q,\boldsymbol{T-\tau})\cdot\varepsilon_{kl}(\boldsymbol{r}_Q,\boldsymbol{\tau})d\tau}$$

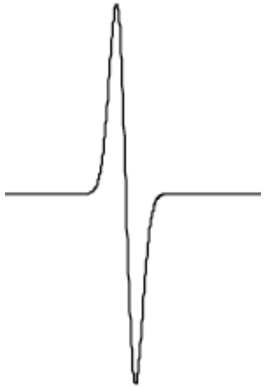
Backward wavefield

Forward wavefield

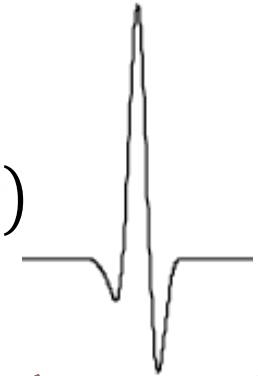


Reversed Forward Wavefield

$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$

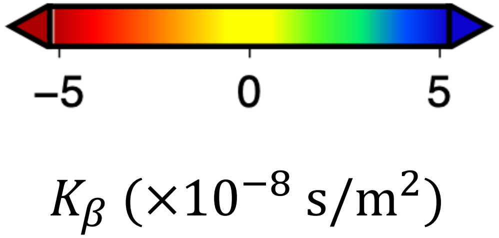


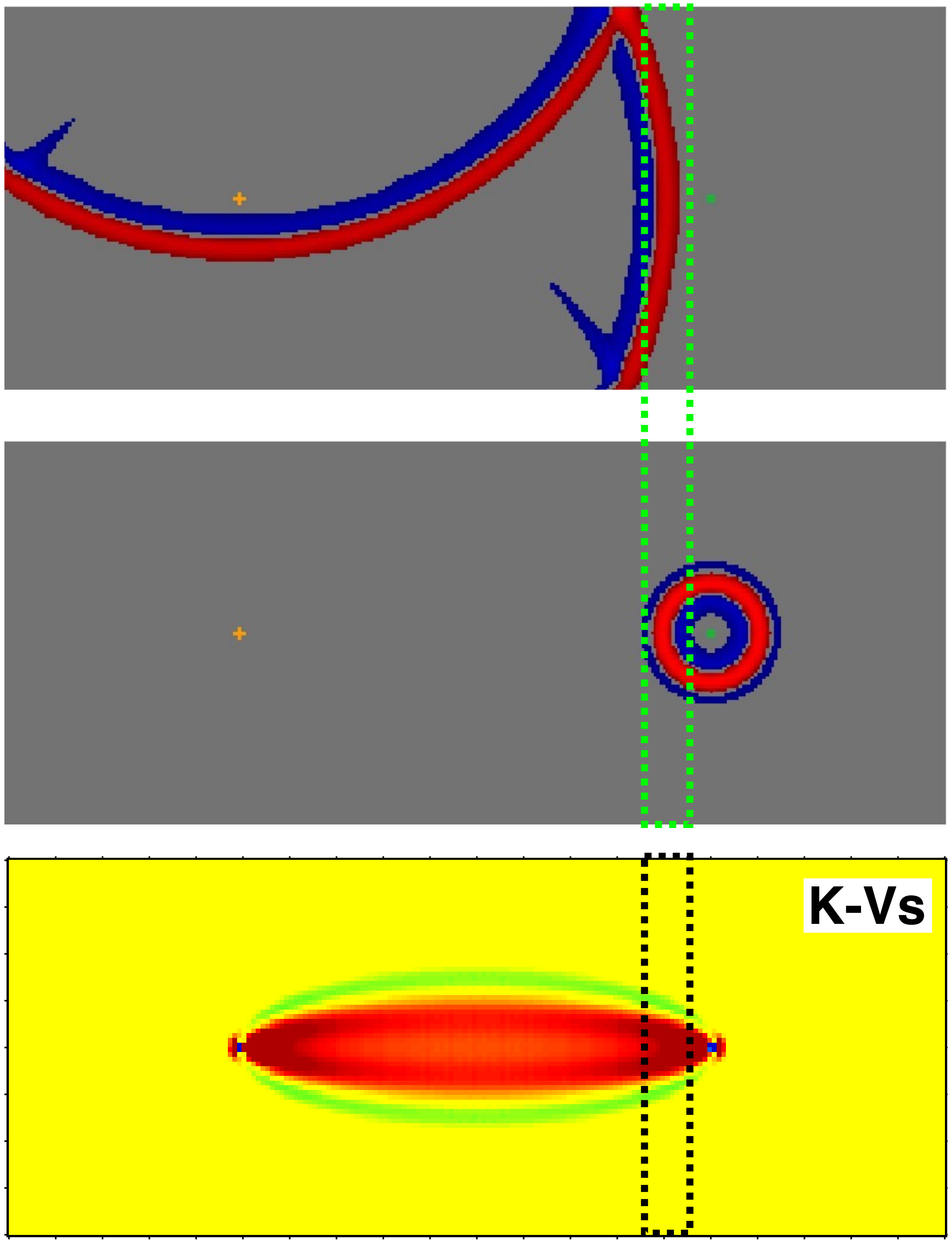
Adjoint Wavefield $\boldsymbol{u}^{\dagger}(\boldsymbol{r}_Q, \tau)$



$$f^{\dagger}(\boldsymbol{r}_R, t) = J^{T_n}(\boldsymbol{r}_R, T-t) \propto -\dot{u}_n(\boldsymbol{r}_R, T-t)$$

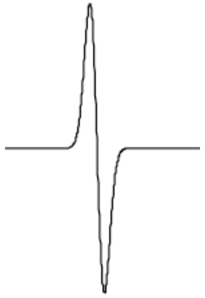
Vs Kernel



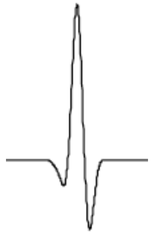


Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$



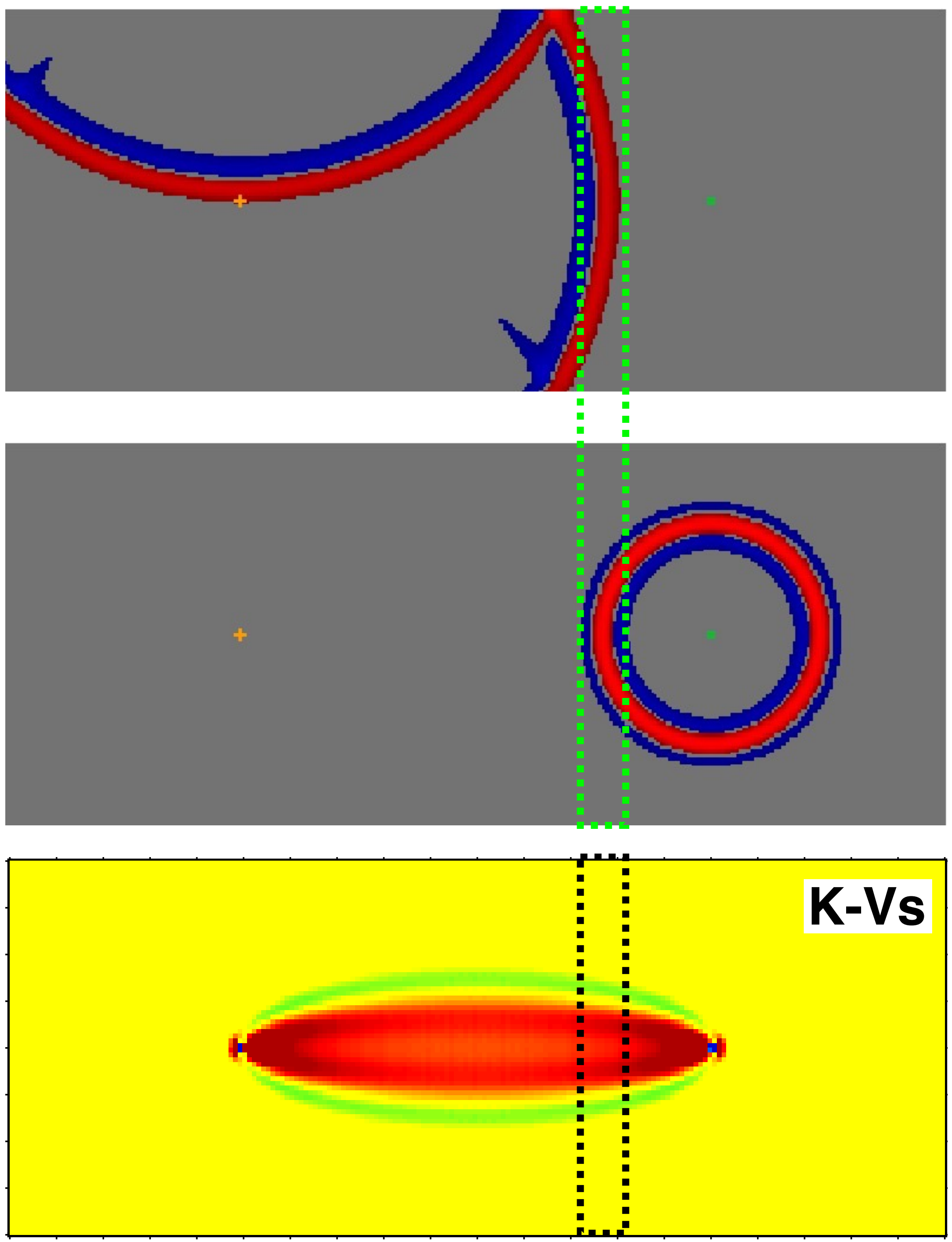
Adjoint Wavefield $\boldsymbol{u}^{\dagger}(\boldsymbol{r}_Q, \tau)$



$$-\int_0^T \varepsilon_{ij}^{\dagger}(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T-\tau) d\tau$$

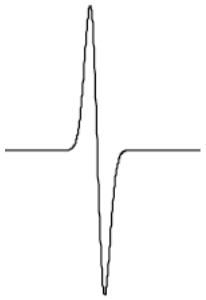
Vs Kernel

K_β ($\times 10^{-8}$ s/m^2)

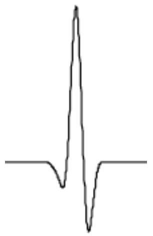


Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$

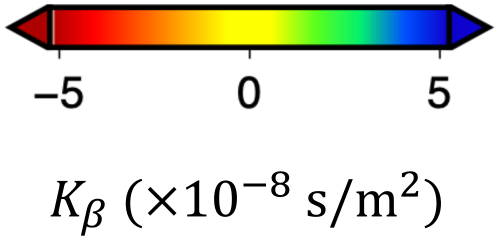


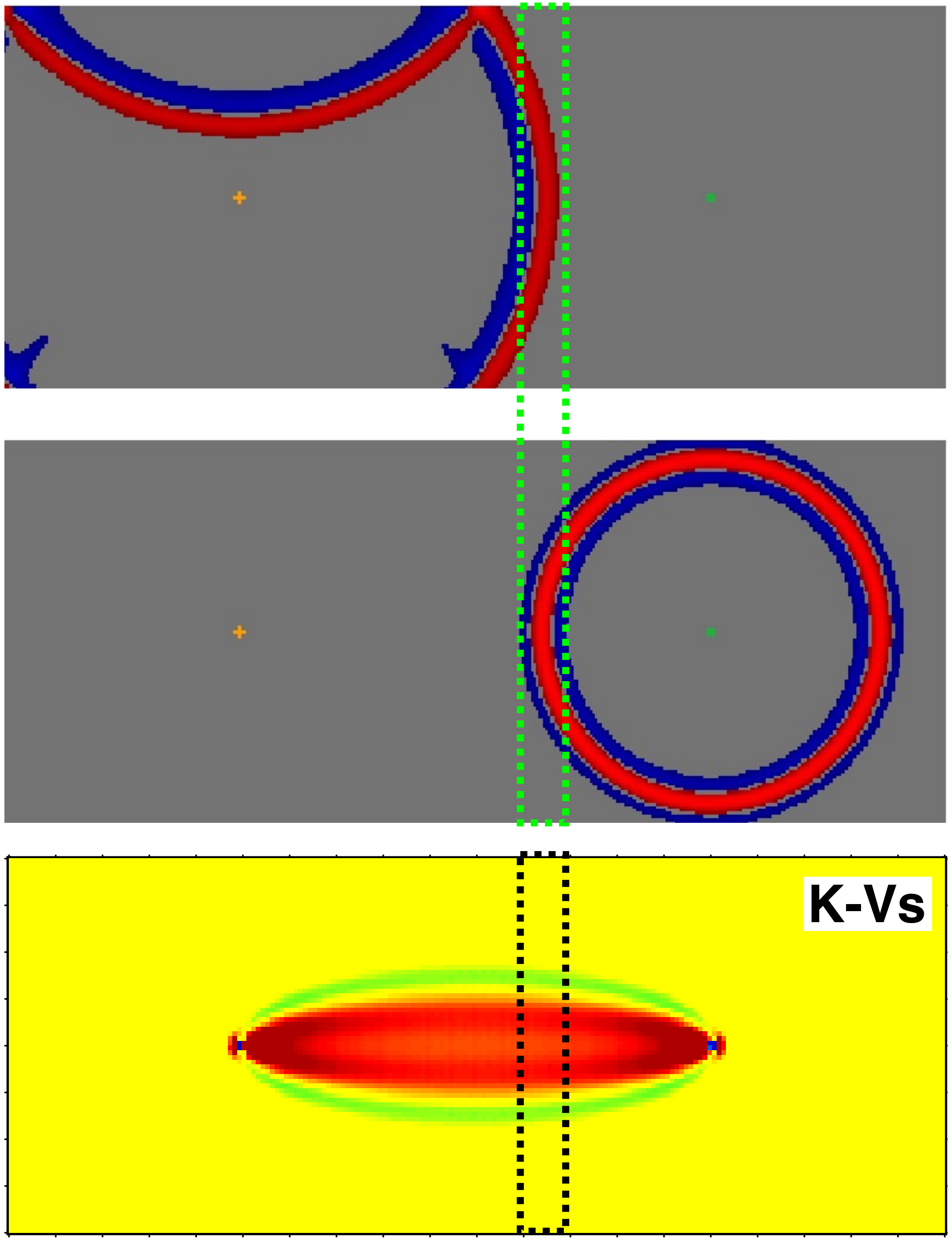
Adjoint Wavefield $\boldsymbol{u}^\dagger(\boldsymbol{r}_Q, \tau)$



$$-\int_0^T \varepsilon^\dagger_{ij}(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T-\tau)\, d\tau$$

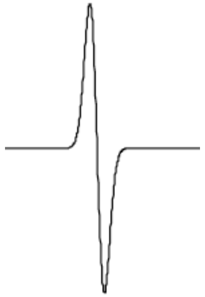
Vs Kernel



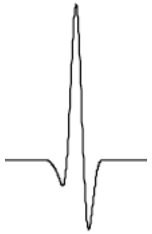


Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$

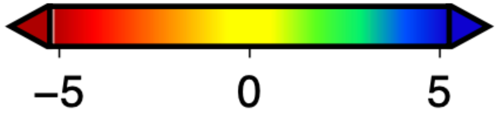


Adjoint Wavefield $\boldsymbol{u}^\dagger(\boldsymbol{r}_Q, \tau)$

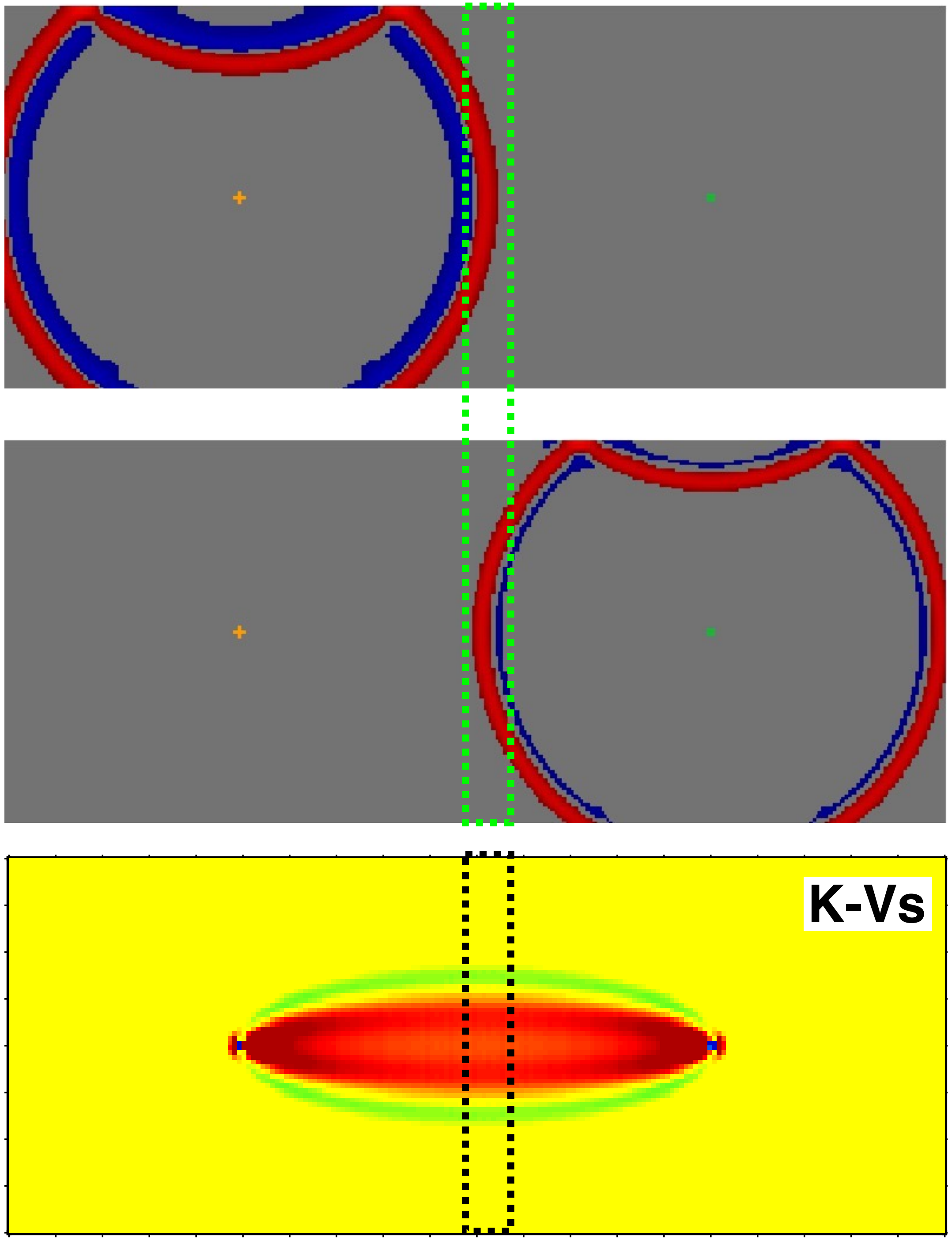


$$-\int_0^T \varepsilon_{ij}^\dagger(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T-\tau)\, d\tau$$

Vs Kernel

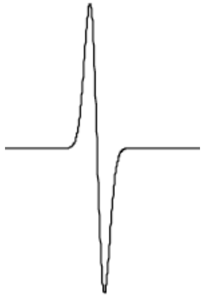


K_β ($\times 10^{-8}$ s/m^2)



Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$



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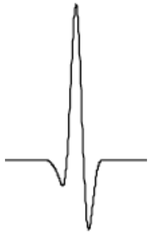

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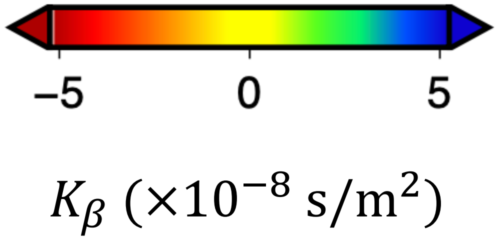
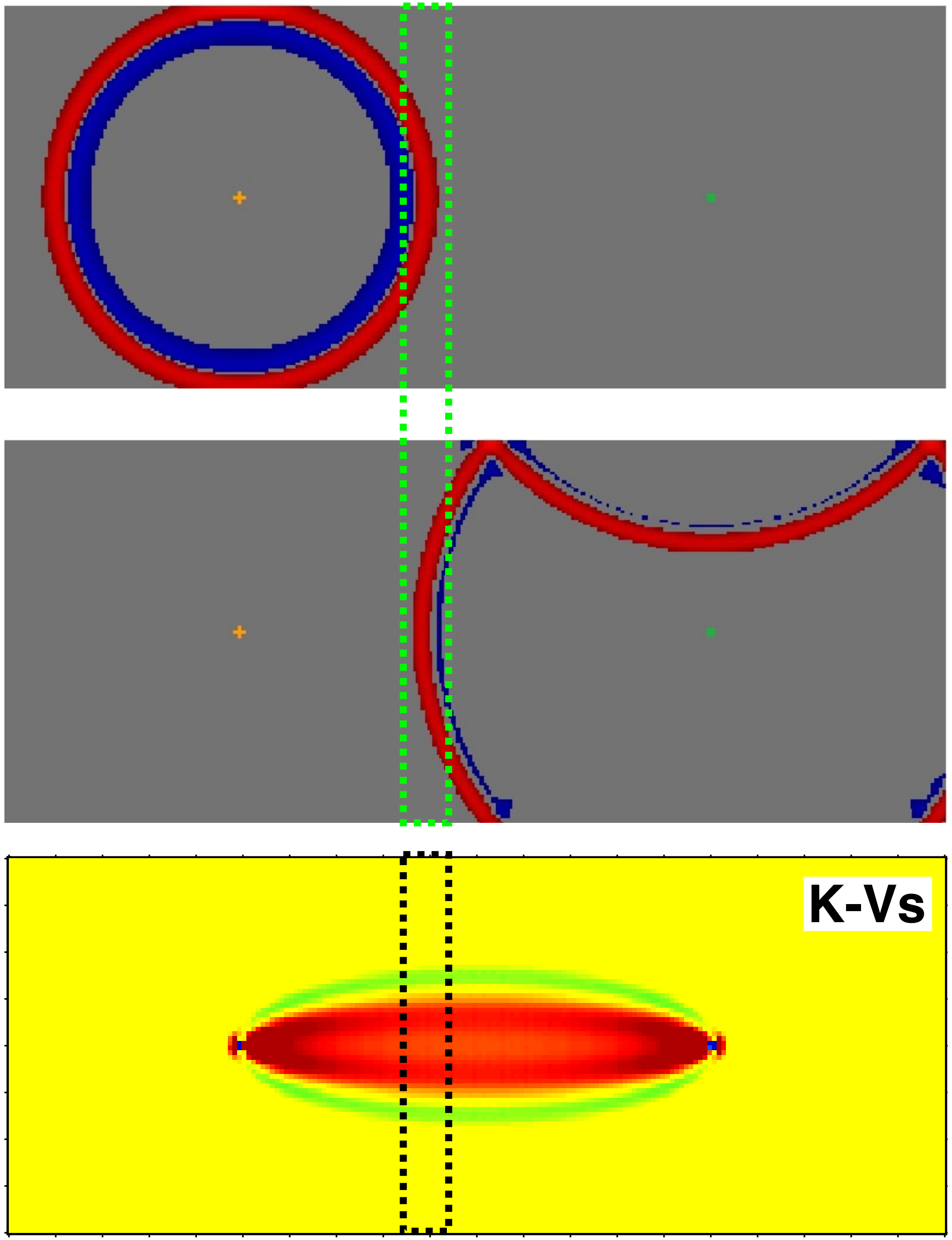
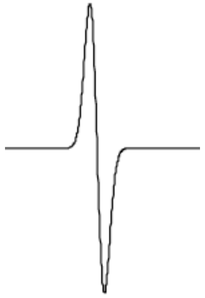
Vs Kernel



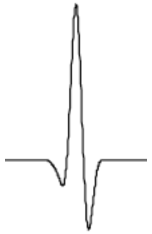


Reversed Forward Wavefield



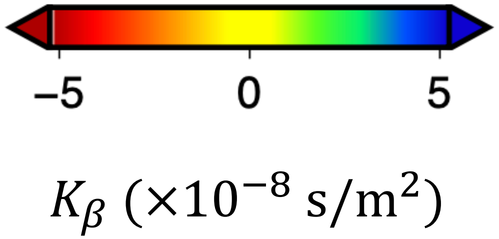
$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$

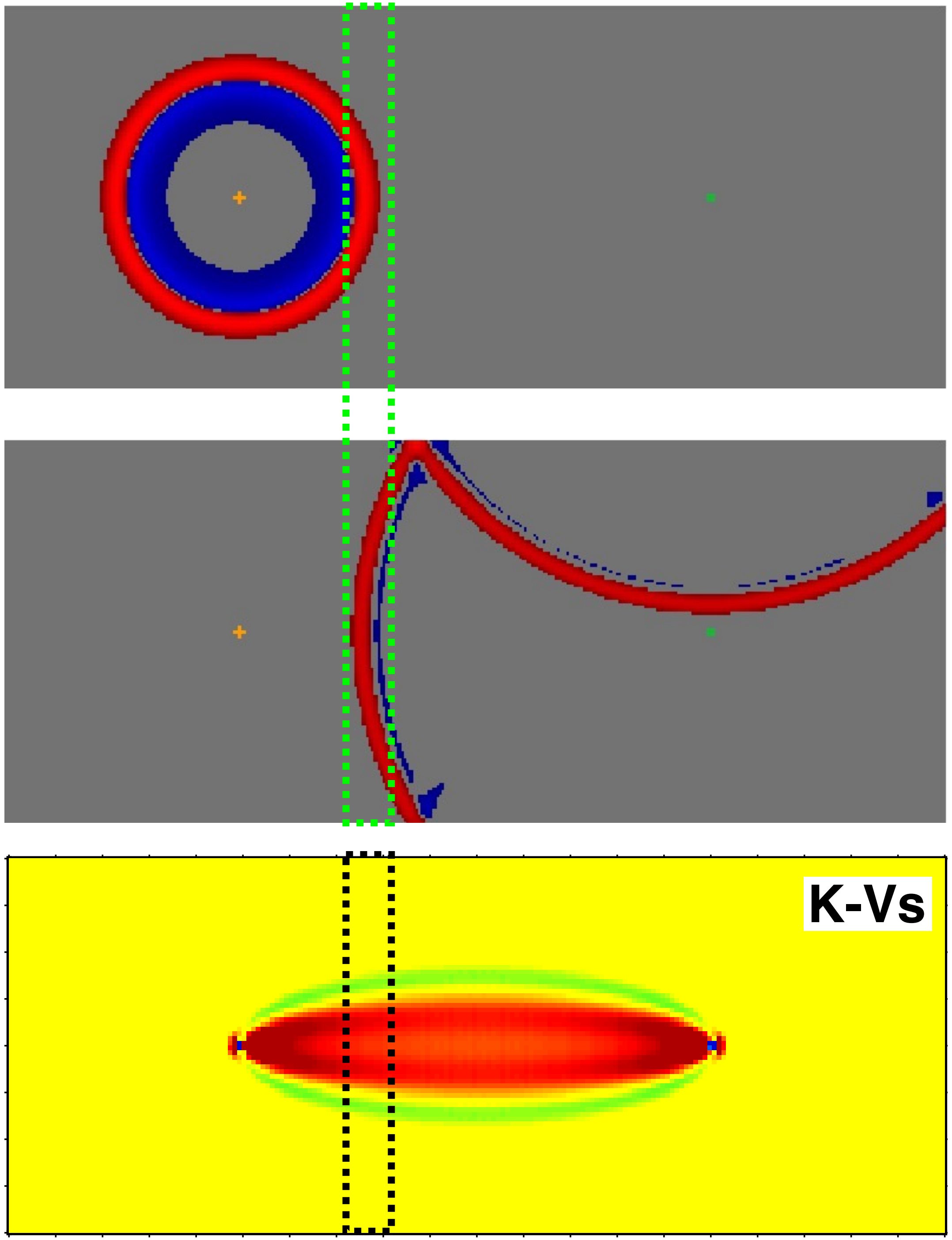
Adjoint Wavefield $\boldsymbol{u}^{\dagger}(\boldsymbol{r}_Q, \tau)$



$$-\int_0^T \varepsilon_{ij}^{\dagger}(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T-\tau) d\tau$$

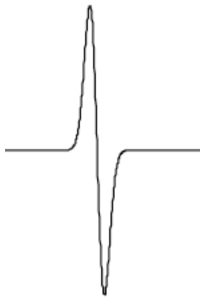
Vs Kernel



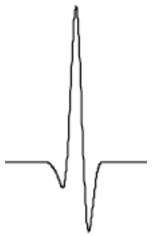


Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$



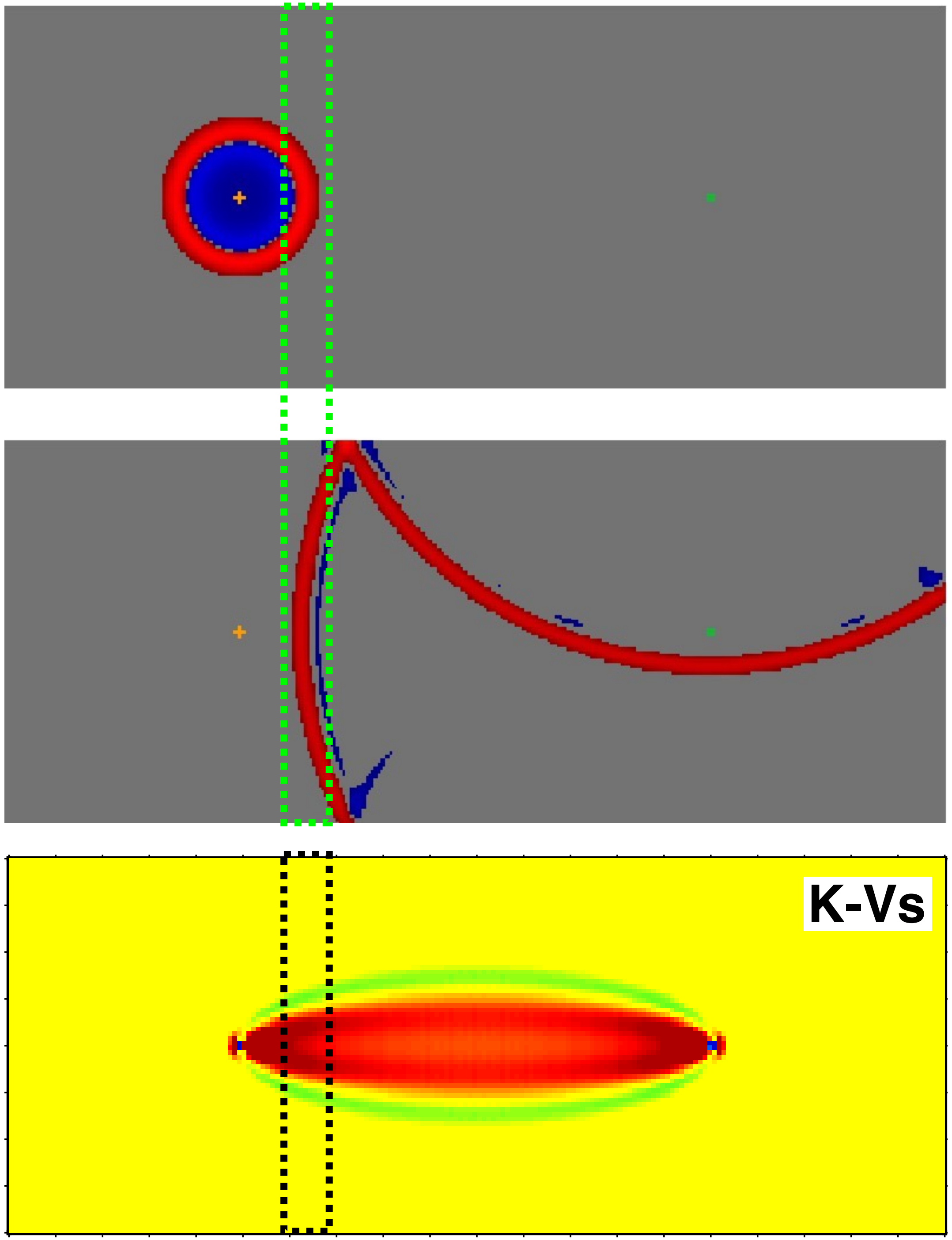
Adjoint Wavefield $\boldsymbol{u}^\dagger(\boldsymbol{r}_Q, \tau)$



$$-\int_0^T \varepsilon_{ij}^\dagger(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T-\tau) d\tau$$

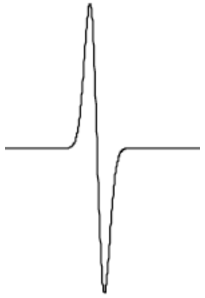
Vs Kernel

K_β ($\times 10^{-8}$ s/m^2)

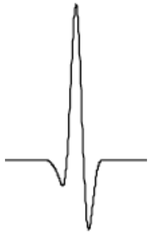


Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T - \tau)$$

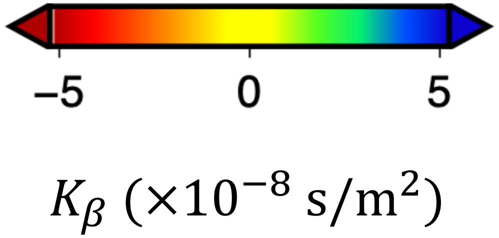


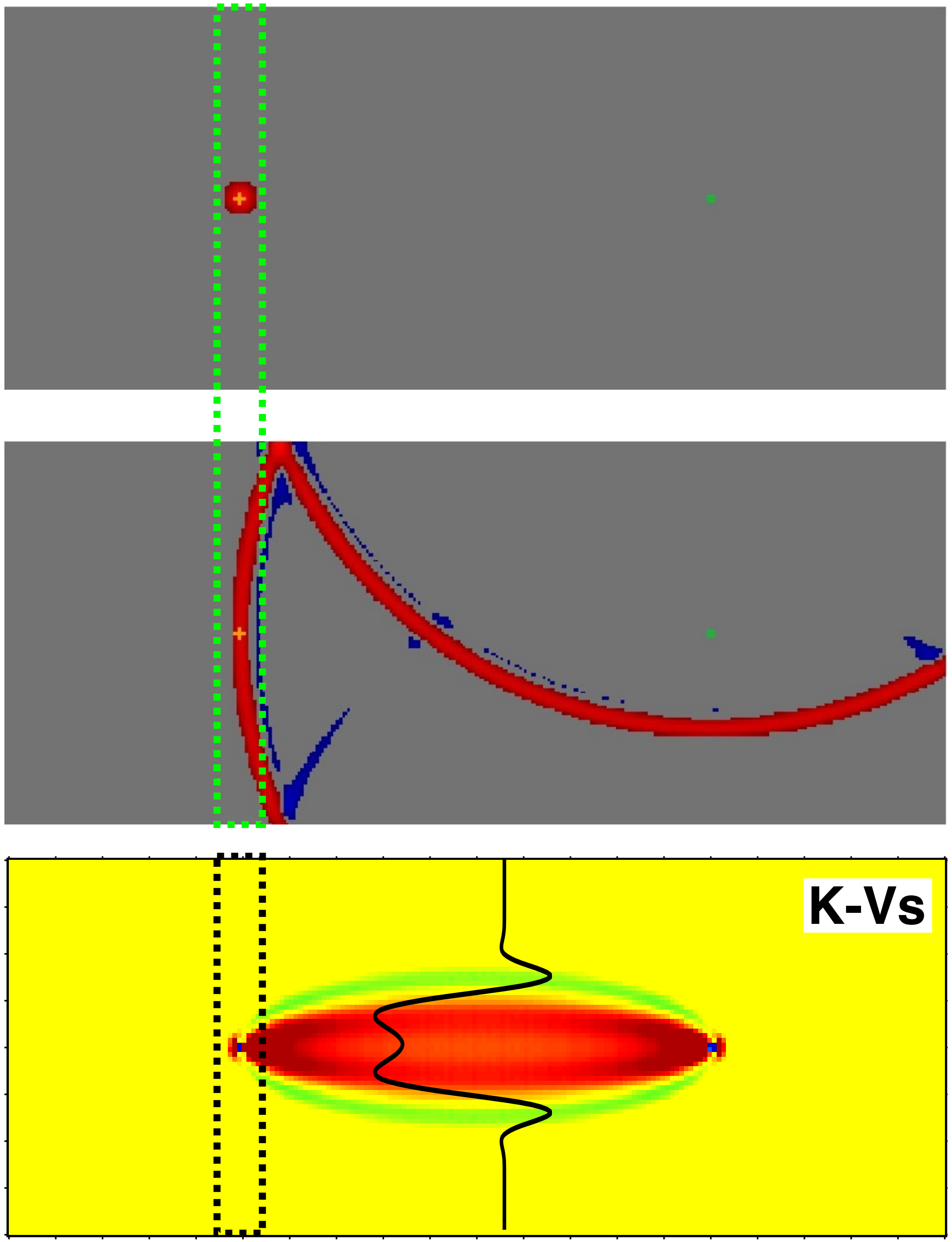
Adjoint Wavefield $\boldsymbol{u}^{\dagger}(\boldsymbol{r}_Q, \tau)$



$$-\int_0^T \varepsilon^{\dagger}_{ij}(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T - \tau) d\tau$$

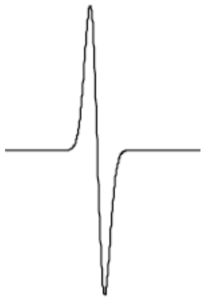
Vs Kernel



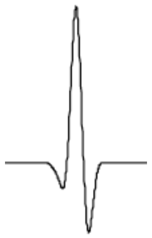


Reversed Forward Wavefield

$$\boldsymbol{u}(\boldsymbol{r}_Q, T-\tau)$$

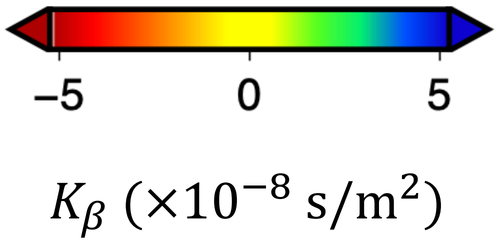


Adjoint Wavefield $\boldsymbol{u}^{\dagger}(\boldsymbol{r}_Q, \tau)$



$$-\int_0^T \varepsilon_{ij}^{\dagger}(\boldsymbol{r}_Q, \tau) \cdot \varepsilon_{kl}(\boldsymbol{r}_Q, T-\tau)\, d\tau$$

Vs Kernel



Rays VS Waves

(A debate between MIT and Princeton)

Global *P* and *PP* traveltime tomography: rays versus waves

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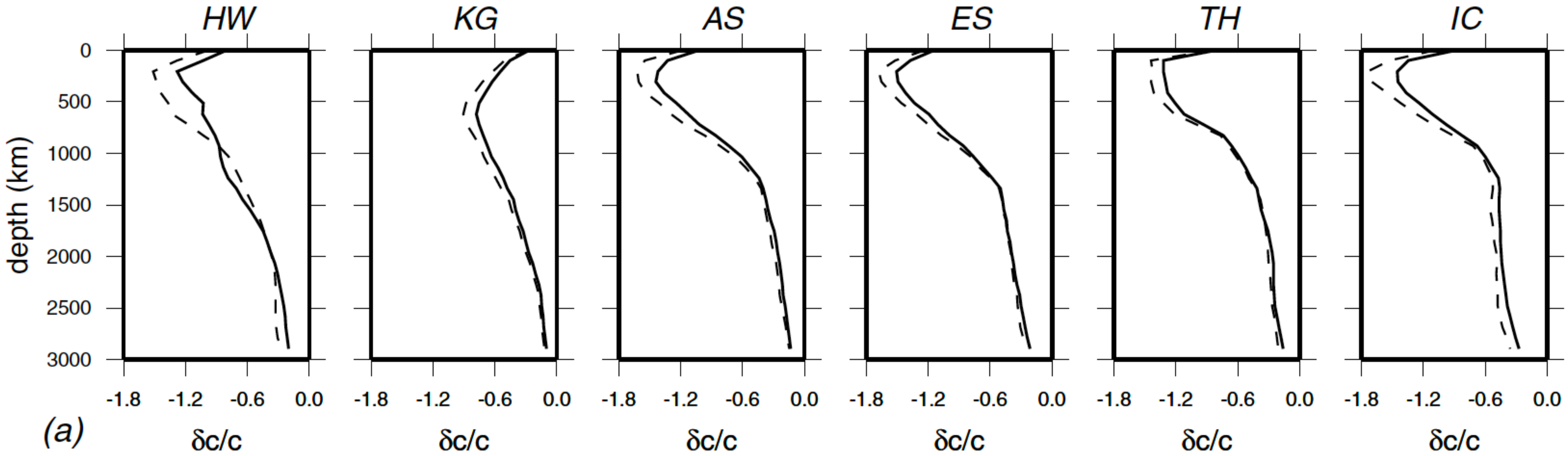
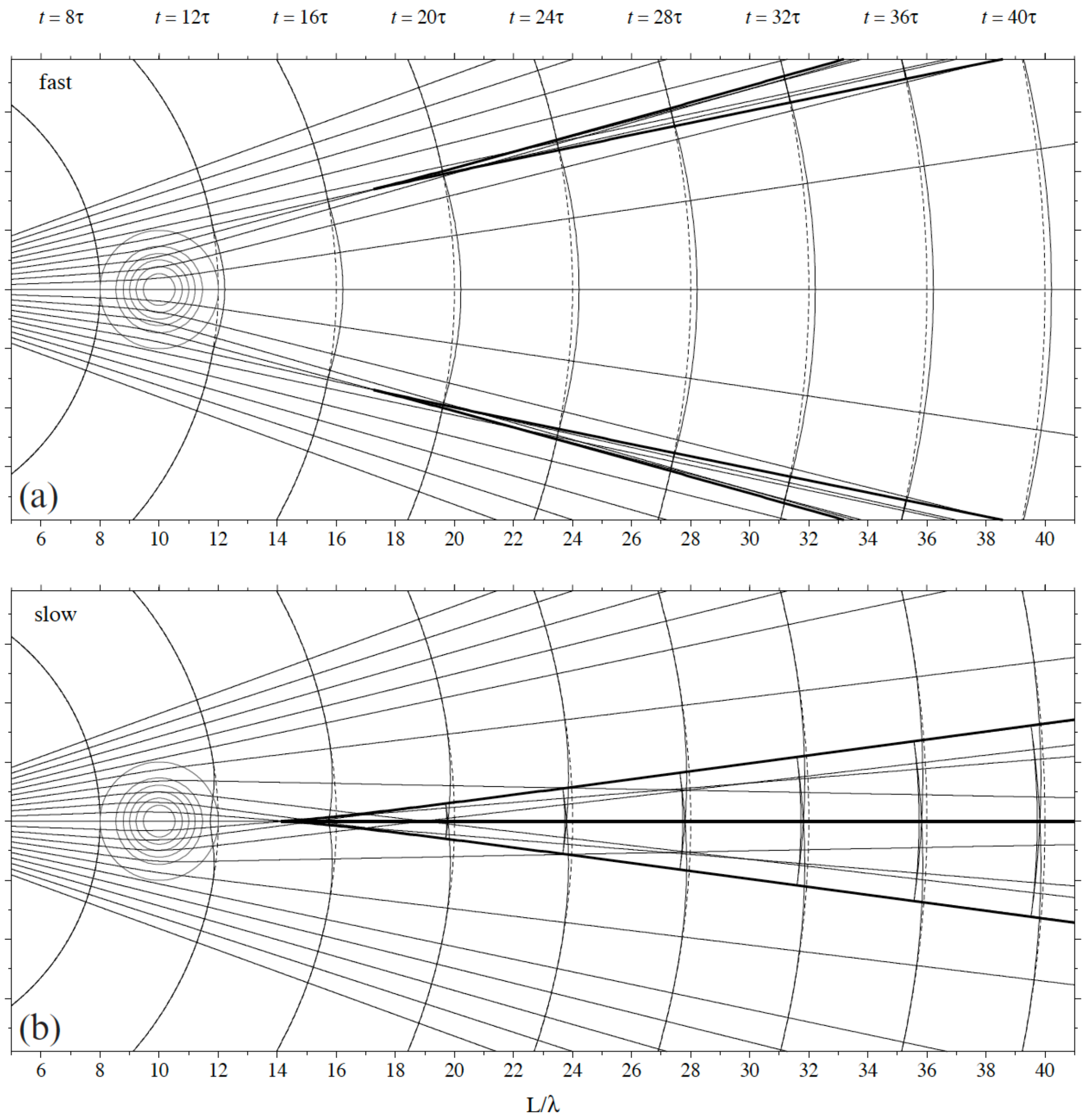
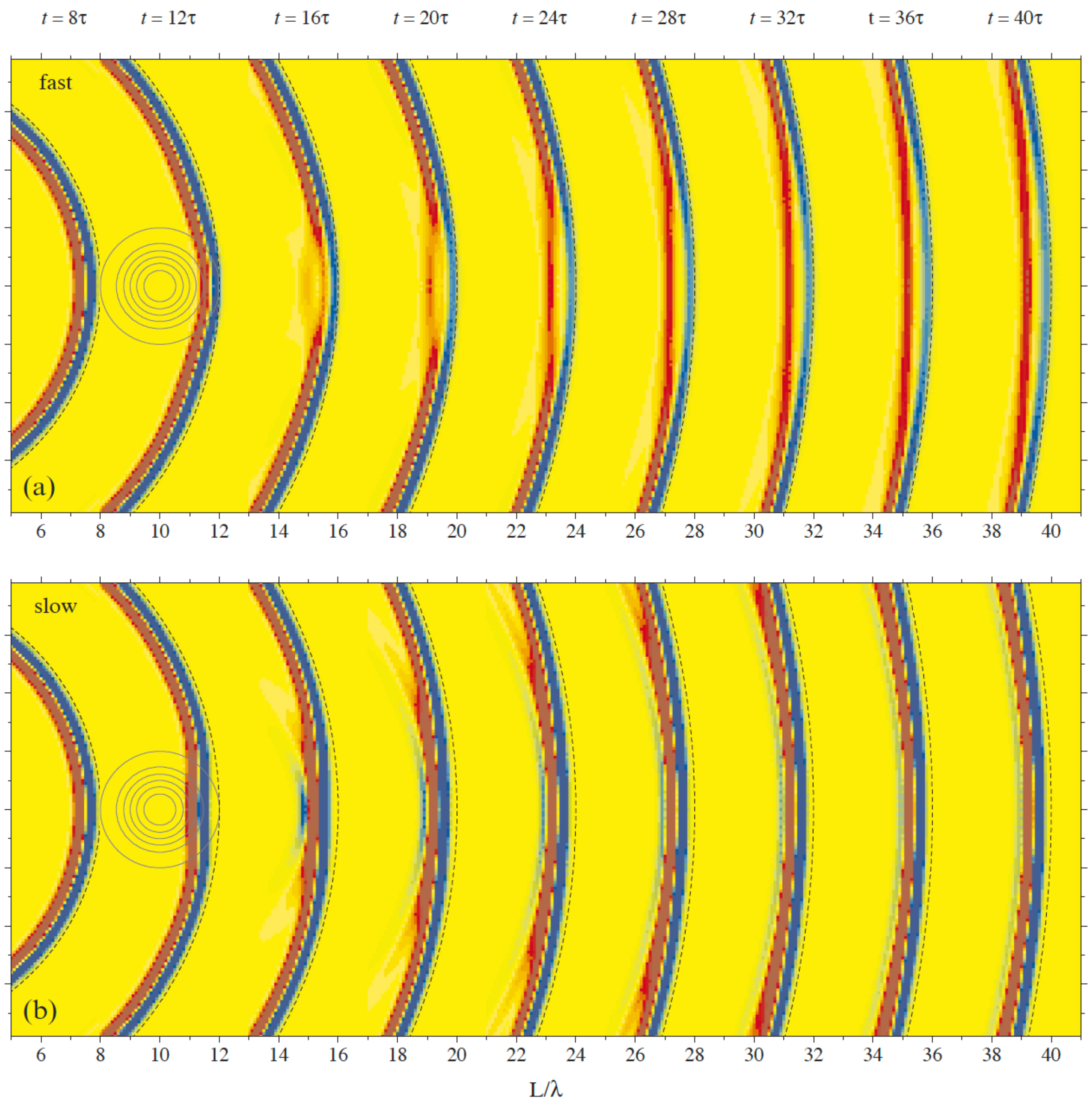


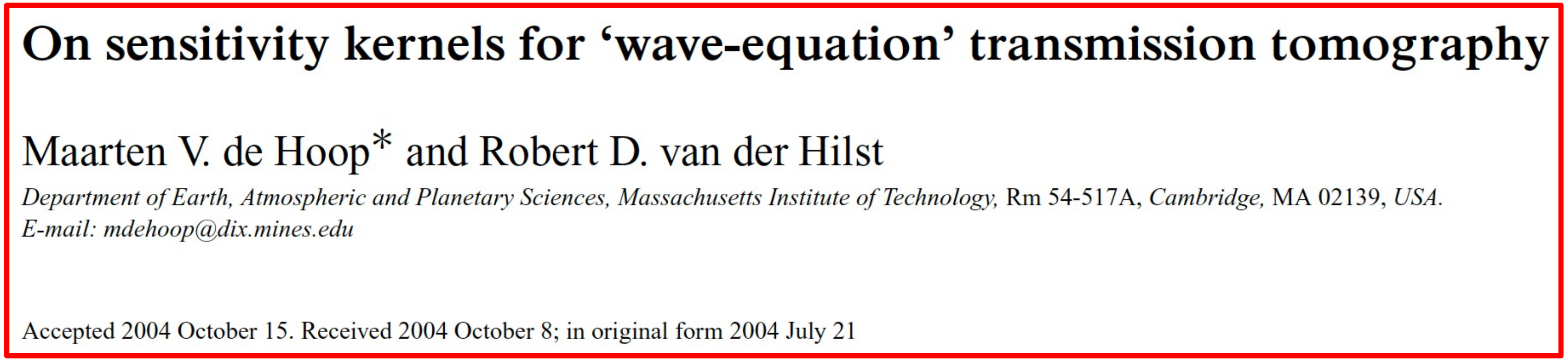
Mantle Plumes: Finite frequency tomography gives larger velocity perturbation

Wavefront Healing Effects

(Hung et al., 2001)







On sensitivity kernels for ‘wave-equation’ transmission tomography

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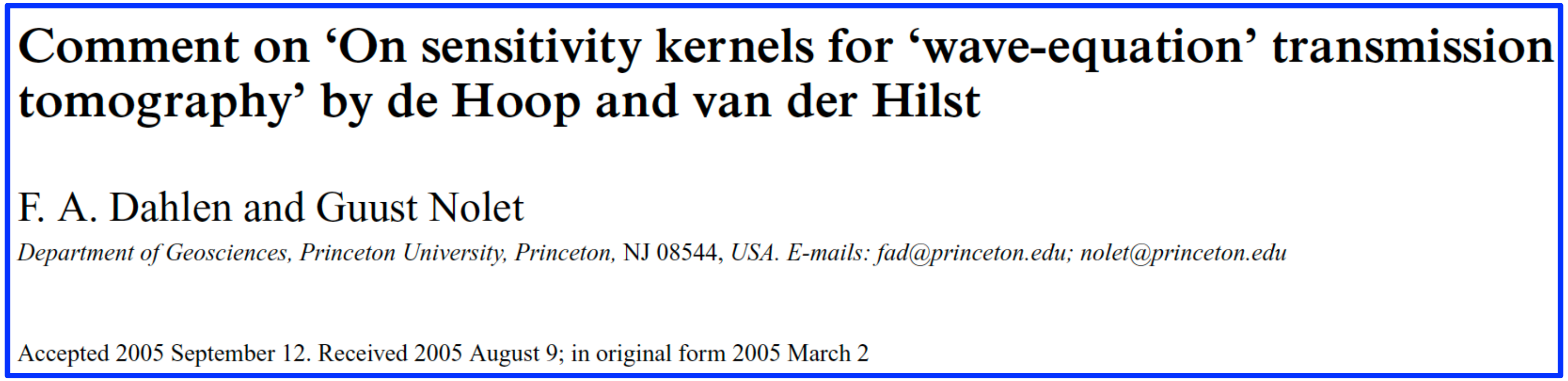

Comment on ‘On sensitivity kernels for ‘wave-equation’ transmission tomography’ by de Hoop and van der Hilst

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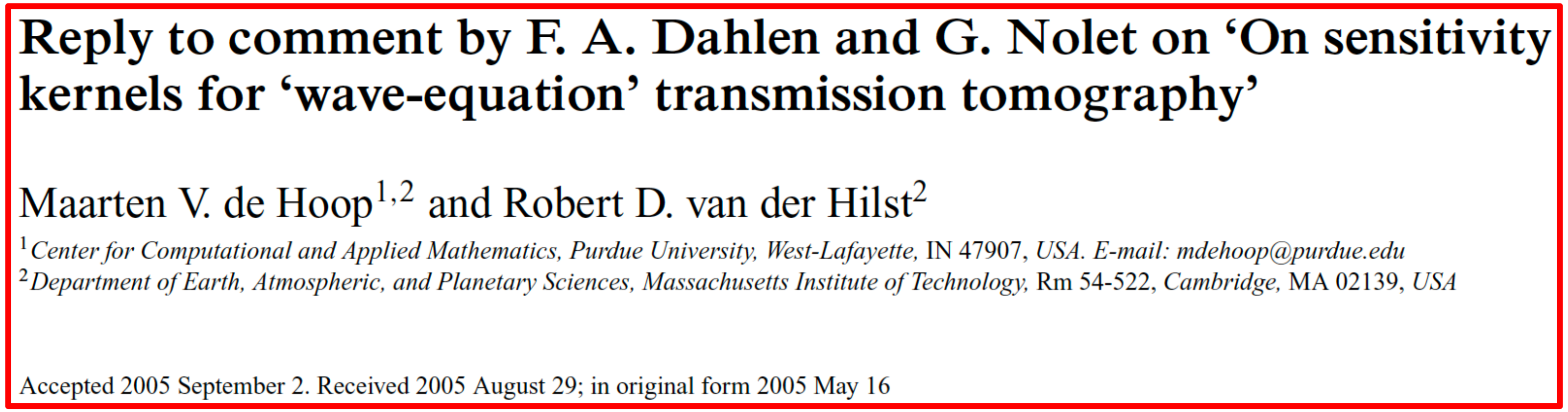

Reply to comment by F. A. Dahlen and G. Nolet on ‘On sensitivity kernels for ‘wave-equation’ transmission tomography’

Maarten V. de Hoop[1,2] and Robert D. van der Hilst[2]
[1]*Center for Computational and Applied Mathematics, Purdue University, West-Lafayette,* IN 47907, *USA. E-mail: mdehoop@purdue.edu*
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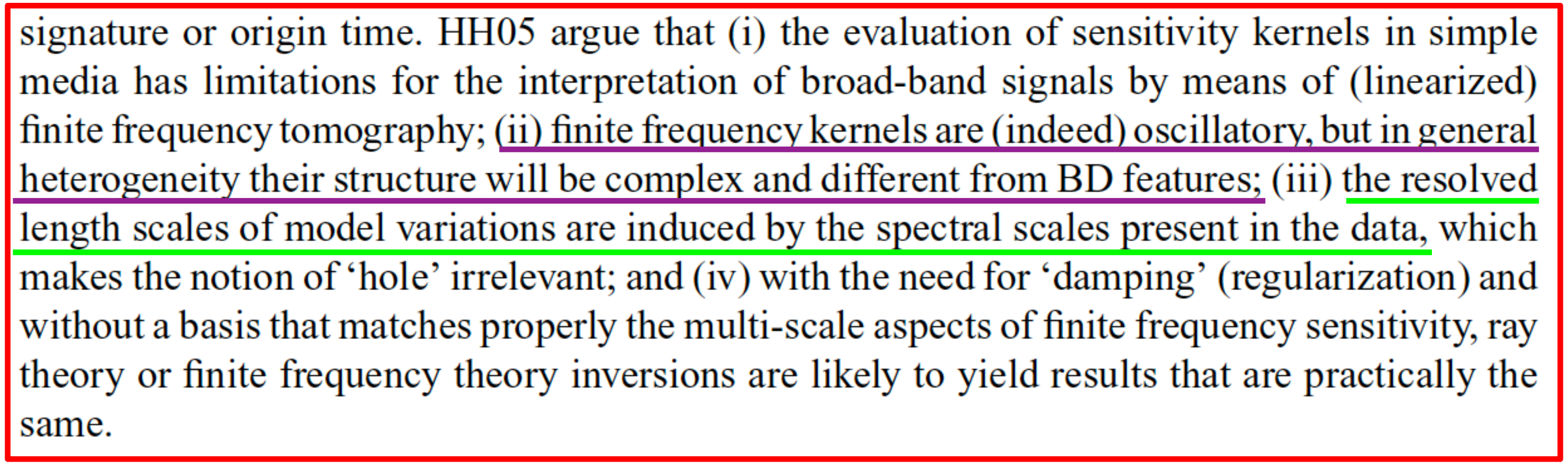
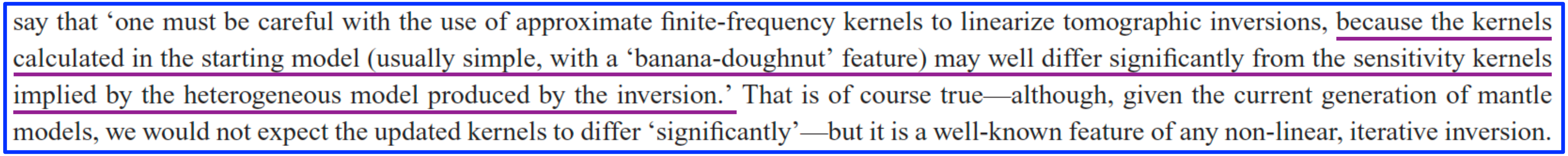
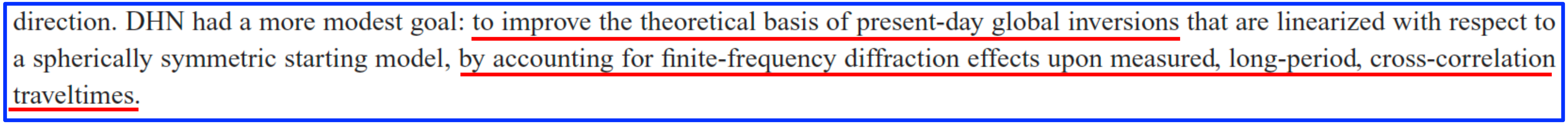
A “Hole” in the Kernel?



signature or origin time. HH05 argue that (i) the evaluation of sensitivity kernels in simple media has limitations for the interpretation of broad-band signals by means of (linearized) finite frequency tomography; (ii) finite frequency kernels are (indeed) oscillatory, but in general heterogeneity their structure will be complex and different from BD features; (iii) the resolved length scales of model variations are induced by the spectral scales present in the data, which makes the notion of ‘hole’ irrelevant; and (iv) with the need for ‘damping’ (regularization) and without a basis that matches properly the multi-scale aspects of finite frequency sensitivity, ray theory or finite frequency theory inversions are likely to yield results that are practically the same.



say that ‘one must be careful with the use of approximate finite-frequency kernels to linearize tomographic inversions, because the kernels calculated in the starting model (usually simple, with a ‘banana-doughnut’ feature) may well differ significantly from the sensitivity kernels implied by the heterogeneous model produced by the inversion.’ That is of course true—although, given the current generation of mantle models, we would not expect the updated kernels to differ ‘significantly’—but it is a well-known feature of any non-linear, iterative inversion.



direction. DHN had a more modest goal: to improve the theoretical basis of present-day global inversions that are linearized with respect to a spherically symmetric starting model, by accounting for finite-frequency diffraction effects upon measured, long-period, cross-correlation traveltimes.