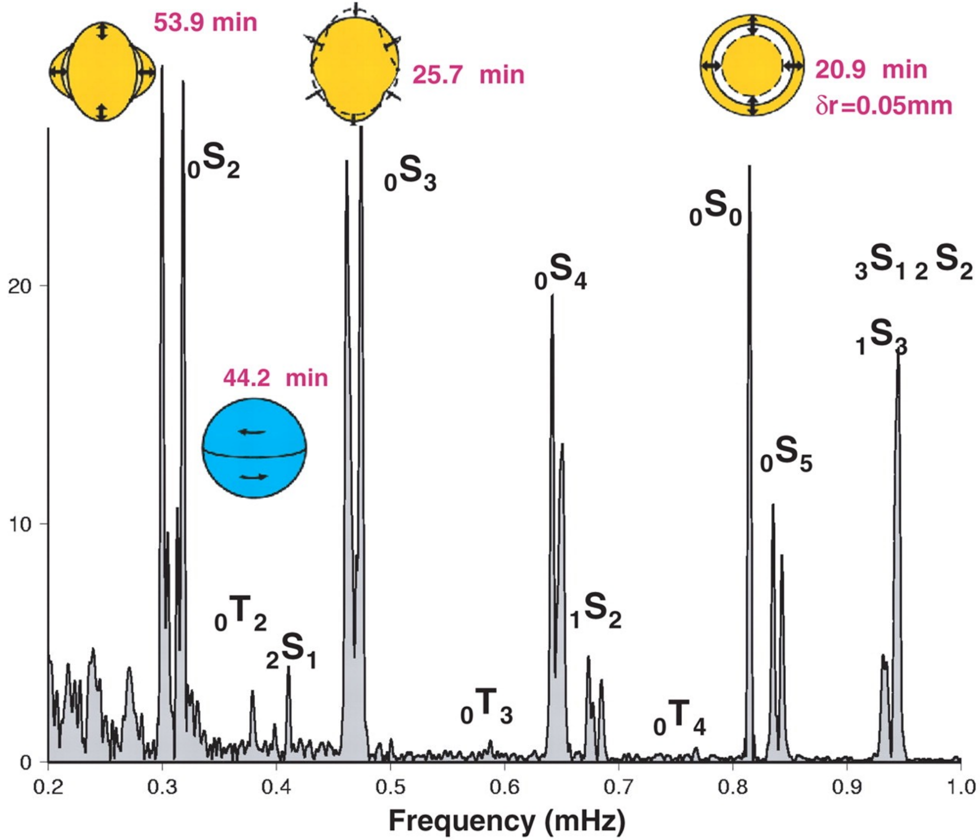
Earth's Free Oscillations (Normal Modes)

TA Lecture for GP 238, Spring 2023

Ji, Qing May 18



Cover figure from:

Park *et al.* (2005). Earth's Free Oscillations Excited by the 26 December 2004 Sumatra-Andaman Earthquake. *Science.*

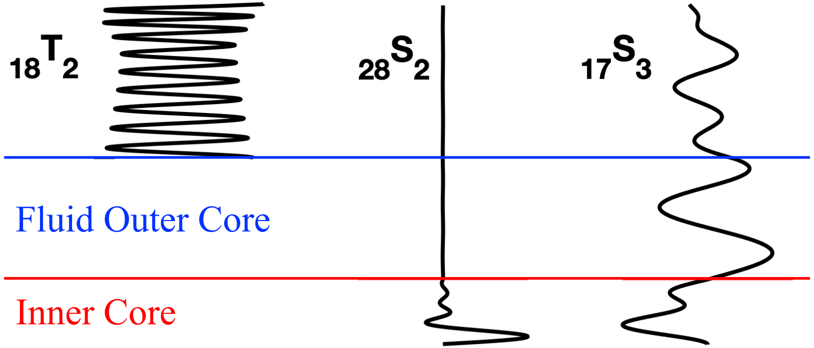
Our Earth can vibrate as a whole, just like a ringing bell!

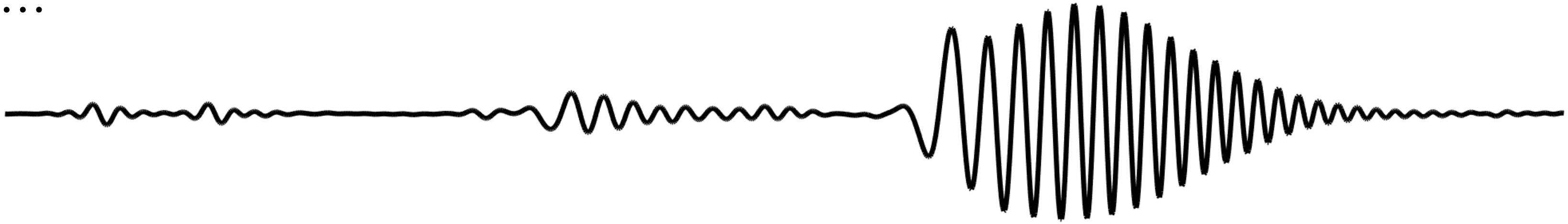
With much exaggeration of course

After large earthquakes or explosions, free oscillations are observed just like the bell after being struck

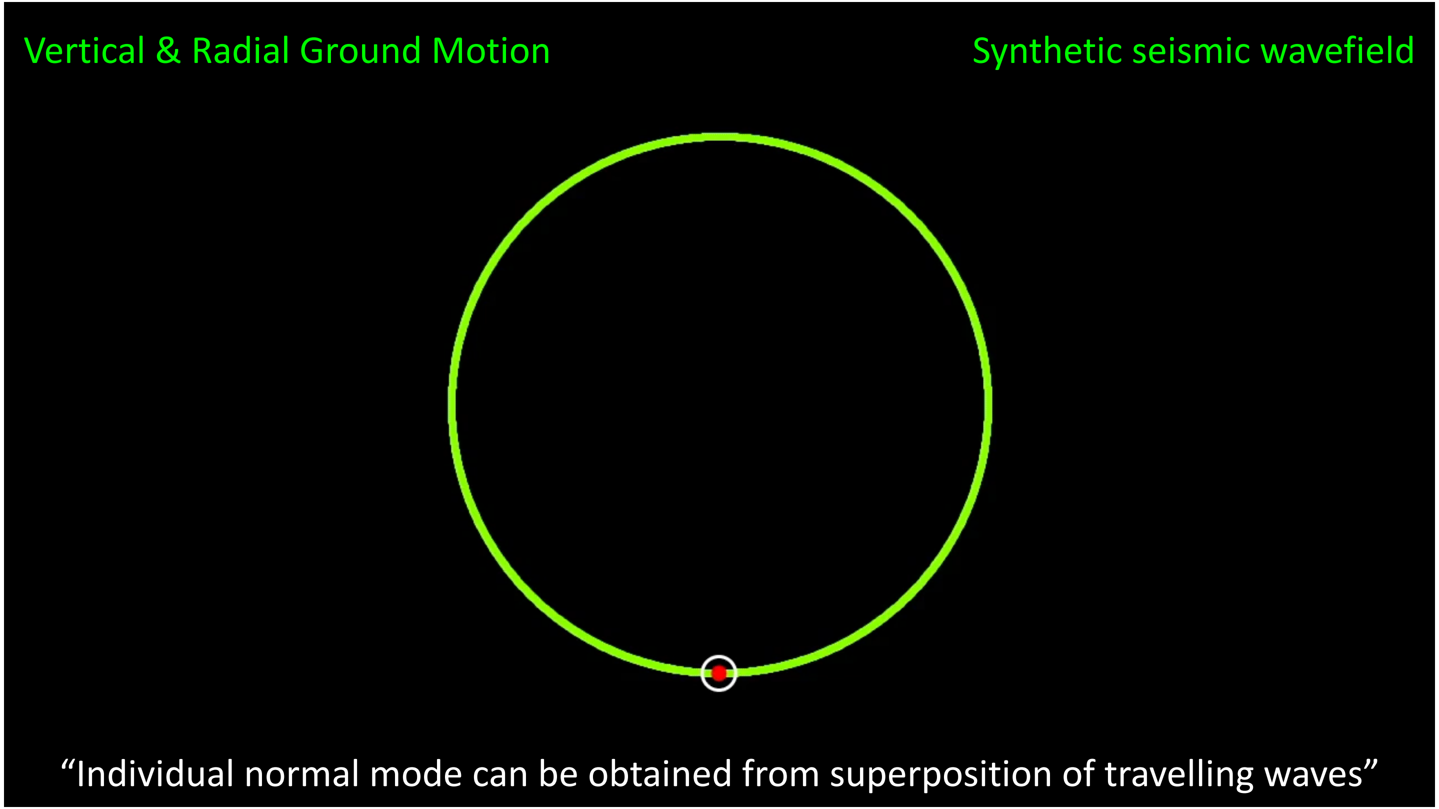
These normal modes can be applied to

- study the (deep) interior of the Earth
- compute synthetic seismic waveforms
- provide info on earthquake size & duration
- ……



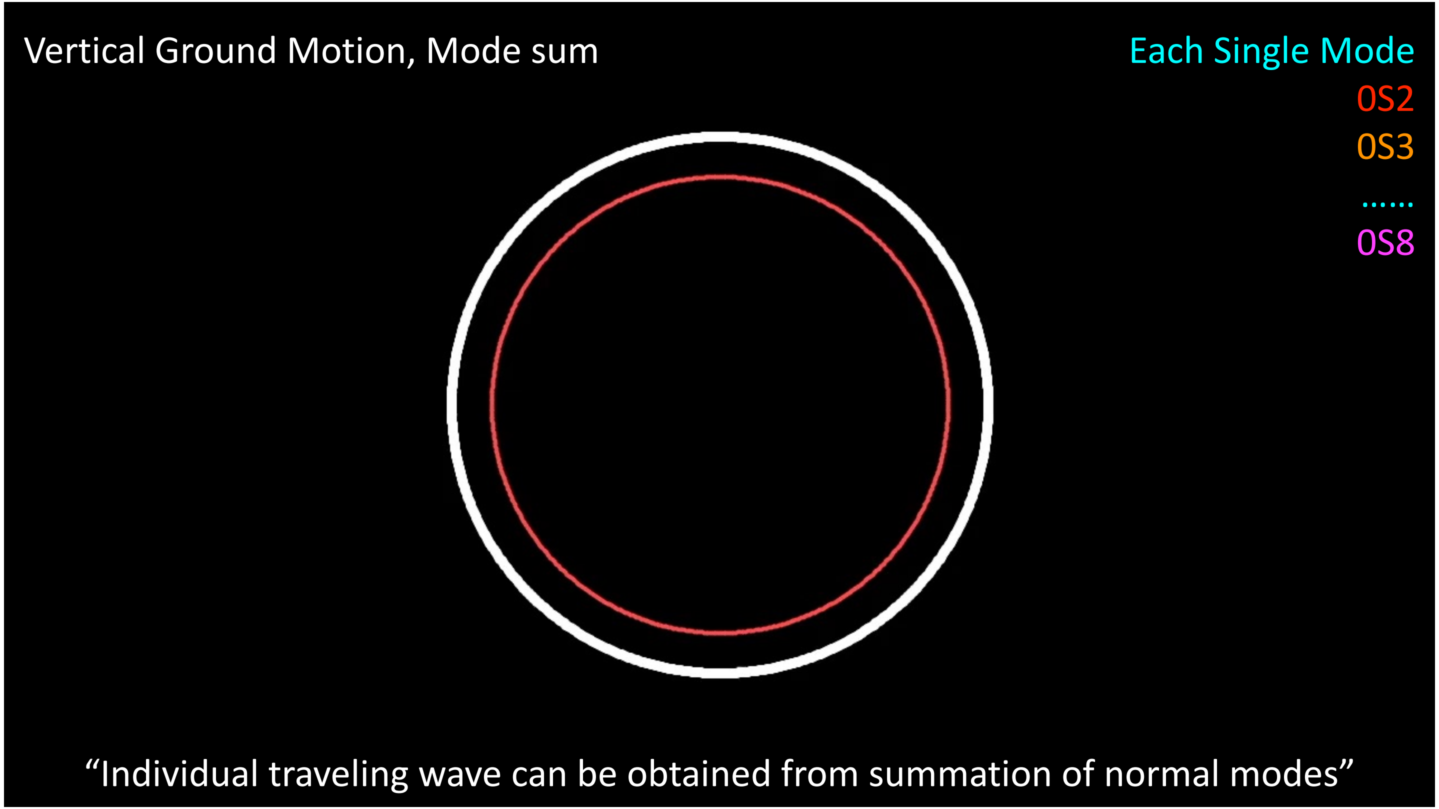


Traveling wave ---> Standing wave



Video courtesy of Celia Eddy and Joshua Russell from Seismic Sound Lab at Lamont-Doherty Earth Observatory

Standing wave ----> Traveling wave



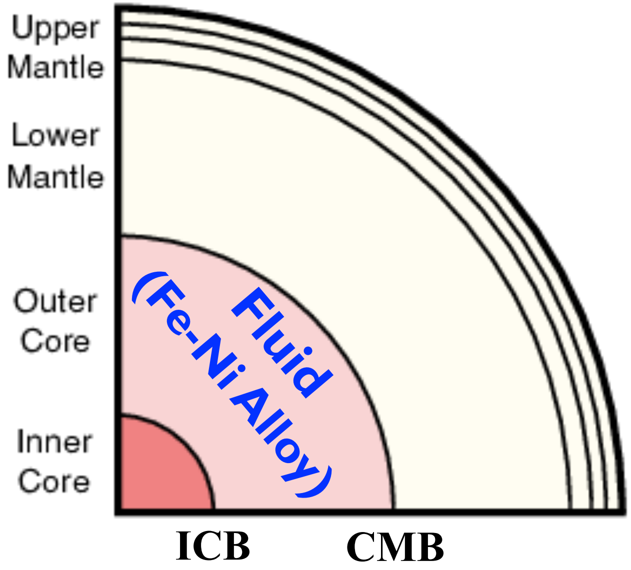
Video courtesy of Celia Eddy and Joshua Russell from Seismic Sound Lab at Lamont-Doherty Earth Observatory

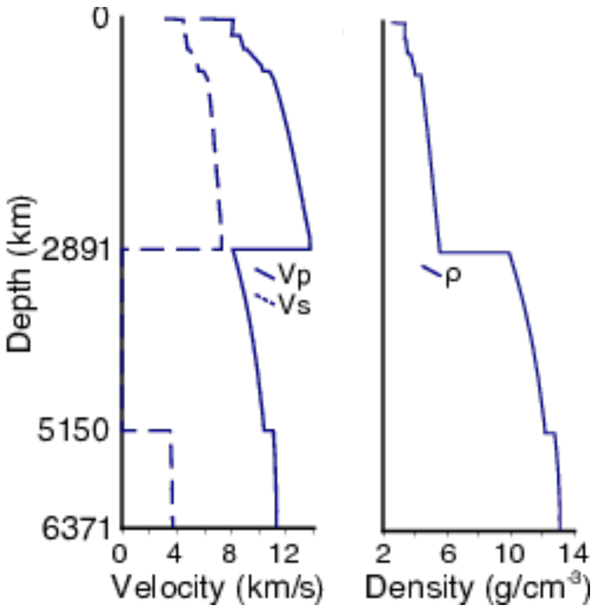
Normal Modes in SNREI Earth Model

SNREI: Spherically symmetric, Non-Rotating, Elastic and Isotropic

Material properties only vary in radial direction

Preliminary Reference Earth Model (PREM)





Dziewonski & Anderson (1981)

Obtained by joint inversion of:

- ✓ Body wave travel times
- ✓ Periods of normal modes (~ 900 modes)

Figure from CIDER

In this lecture ……

Traveling wave & Standing wave: Ray-mode duality

- ✓ Individual normal mode can be obtained from superposition of travelling waves
- ✓ Individual traveling wave (P, S, surface waves …) can be obtained from summation of normal modes

We can get a sense of this duality by visualizing radial eigenfunction

Excitation of normal modes: Calculate Green's function, synthetic seismogram

(Complete Basis!)

$$\mathbf{G}(\mathbf{r}, t; \mathbf{r}_0) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\sin({}_n\omega_l t)}{{}_n\omega_l} \, {}_n\mathbf{s}_{lm}(\mathbf{r}) \, {}_n\mathbf{s}_{lm}(\mathbf{r}_0), \qquad t > 0.$$

Procedure to solve normal modes

- Linearized equations of motion in frequency domain, SNREI model

$$-\omega^2 \rho \boldsymbol{u} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \nabla (g u_r) + \rho \nabla \phi - \rho (\boldsymbol{\nabla} \cdot \boldsymbol{u}) g \hat{\boldsymbol{r}} = 0$$

Gravitational effect is important at low frequency

Hooke's law:

$$\boldsymbol{\sigma} = K(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{I} + 2\mu \left[\frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla} u^T) - \frac{1}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{I}\right]$$

- Linearized boundary conditions

	Solid-Solid	**Solid-Fluid**	**Free surface**
Kinematic (Displacement $\boldsymbol{u}$)	$[\boldsymbol{u}]_-^+ = \boldsymbol{0}$	$[\hat{\boldsymbol{n}} \cdot \boldsymbol{u}]_-^+ = 0$	———
Dynamic (Normal traction $\boldsymbol{T} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}$)	$[\boldsymbol{T}]_-^+ = \boldsymbol{0}$	$\hat{\boldsymbol{n}}[\hat{\boldsymbol{n}} \cdot \boldsymbol{T}]_-^+ = [\boldsymbol{T}]_-^+ = \boldsymbol{0}$	$\boldsymbol{T} = \boldsymbol{0}$

Procedure to solve normal modes

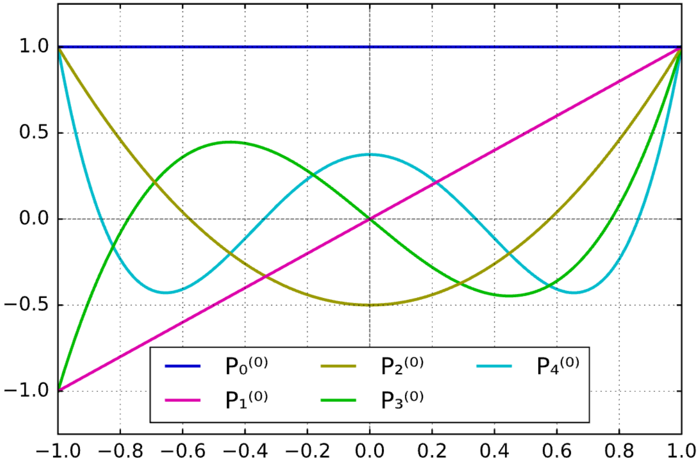
- Expand displacement field $\boldsymbol{u}$ with Vector Spherical Harmonics. For each mode:

$$_n\mathbf{u}_{lm}(r,\theta,\phi) = {}_nU_l(r)\mathbf{P}_{lm}(\theta,\phi) + {}_nV_l(r)\mathbf{B}_{lm}(\theta,\phi) + {}_nW_l(r)\mathbf{C}_{lm}(\theta,\phi)$$

SNREI model leads to **degeneracy in $\boldsymbol{m}$**. Mode frequency and radial function only depend on n and l

Vector Spherical Harmonics: $\mathbf{P}_{lm}(\theta,\phi)$, $\mathbf{B}_{lm}(\theta,\phi)$, $\mathbf{C}_{lm}(\theta,\phi)$

- Related to real-valued spherical harmonics $\mathcal{Y}_{lm}(\theta,\phi)$ and its surface gradient and surface curl
- Product between $P_{lm}(\cos\theta)$ and $\sin(m\phi)$ or $\cos(m\phi)$

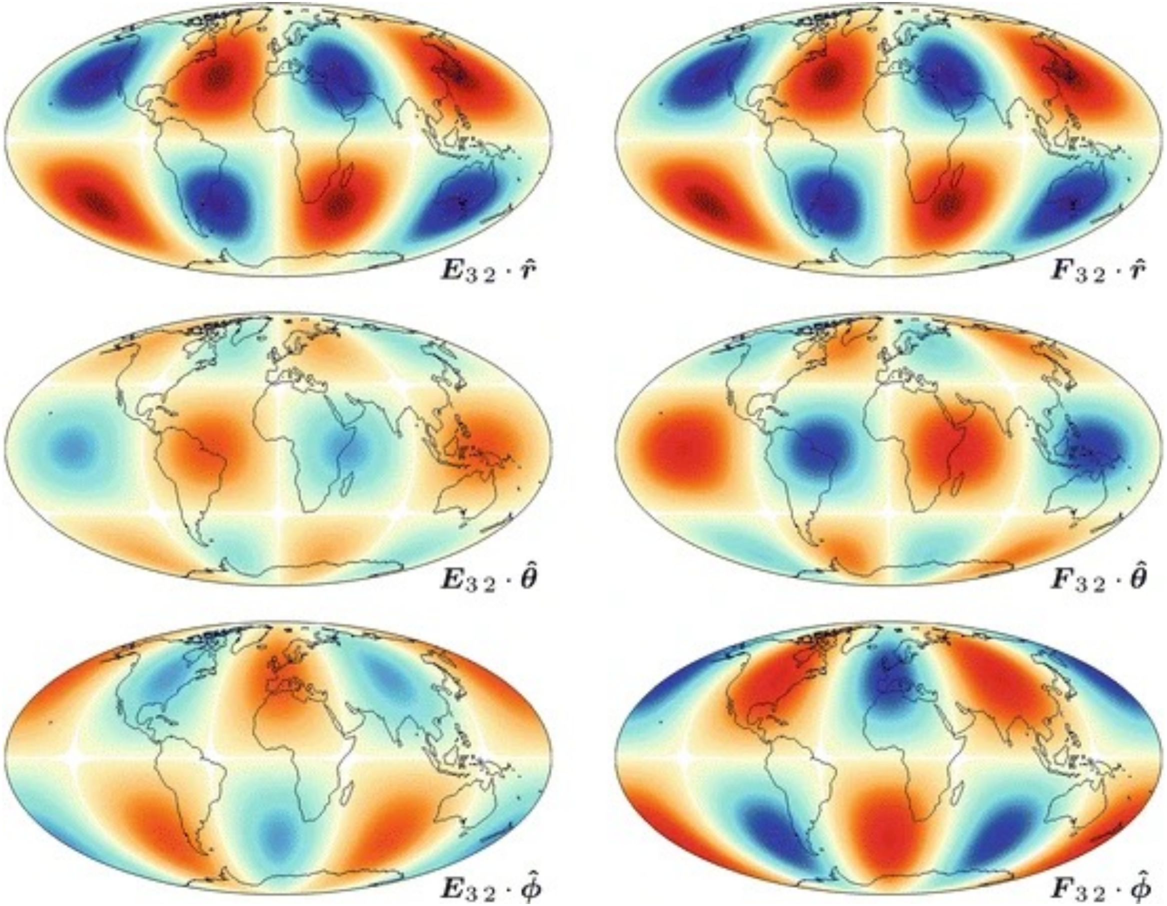


n, l, m indicate number of nodes in the eigenfunctions for radial, latitudinal and longitudinal directions

Plots of associated Legendre polynomials from Wikipedia

n, l, m are related to number of nodes in the eigenfunctions for radial, latitudinal and longitudinal directions

Higher orders indicate smaller wavelength, finer spatial resolution



Plattner, A., Simons, F.J. (2013). Potential-Field Estimation Using Scalar and Vector Slepian Functions at Satellite Altitude. Handbook of Geomathematics.

Procedure to solve normal modes

- ➢ Substitute the expansion into wave equation
- Decouple into **P-SV (spheroidal)** and **SH (toroidal)** systems
- Change problem from a second-order ODE into first-order ODE system by introducing traction vector

$$ {}_n\mathbf{u}_{lm}(r,\theta,\phi) = {}_nU_l(r)\mathbf{P}_{lm}(\theta,\phi) + {}_nV_l(r)\mathbf{B}_{lm}(\theta,\phi) + {}_nW_l(r)\mathbf{C}_{lm}(\theta,\phi) $$

$$ \hat{\mathbf{r}}\cdot\boldsymbol{\sigma} = {}_n\mathbf{T}_{lm}(r,\theta,\phi) = \underbrace{{}_nR_l(r)\mathbf{P}_{lm}(\theta,\phi) + {}_nS_l(r)\mathbf{B}_{lm}(\theta,\phi)}_{\text{P-SV}} + \underbrace{{}_nT_l(r)\mathbf{C}_{lm}(\theta,\phi)}_{\text{SH}} $$

P-SV **SH**

Example: ODEs for SH (toroidal) system

$$ r^{-2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\mu r^2\left(\frac{\mathrm{d}W}{\mathrm{d}r} - r^{-1}W\right)\right] + \mu r^{-1}\left(\frac{\mathrm{d}W}{\mathrm{d}r} - r^{-1}W\right) + \left[\omega^2\rho - (l^2+l-2)\mu r^{-2}\right]W = 0 $$

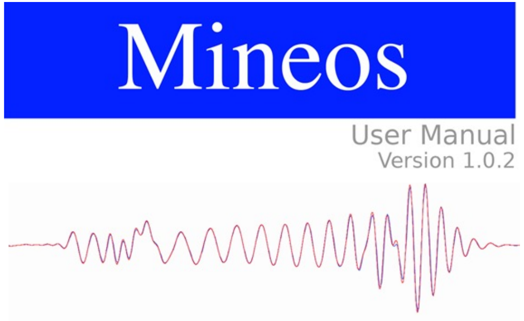
$$ T = \mu\left(\frac{\mathrm{d}W}{\mathrm{d}r} - r^{-1}W\right) $$

$$ \frac{\mathrm{d}W}{\mathrm{d}r} = r^{-1}W + \mu^{-1}T $$

$$ \frac{\mathrm{d}T}{\mathrm{d}r} = \left[-\omega^2\rho + (l^2+l-2)\mu r^{-2}\right]W - 3r^{-1}T $$

Procedure to solve normal modes

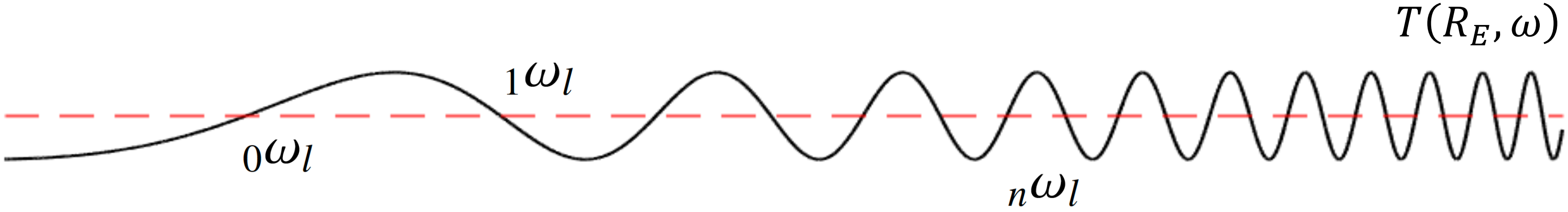
$$\frac{\mathrm{d}W}{\mathrm{d}r} = r^{-1}W + \mu^{-1}T$$
$$\frac{\mathrm{d}T}{\mathrm{d}r} = \left[-\omega^2\rho + (l^2 + l - 2)\mu r^{-2}\right] W - 3r^{-1}T$$

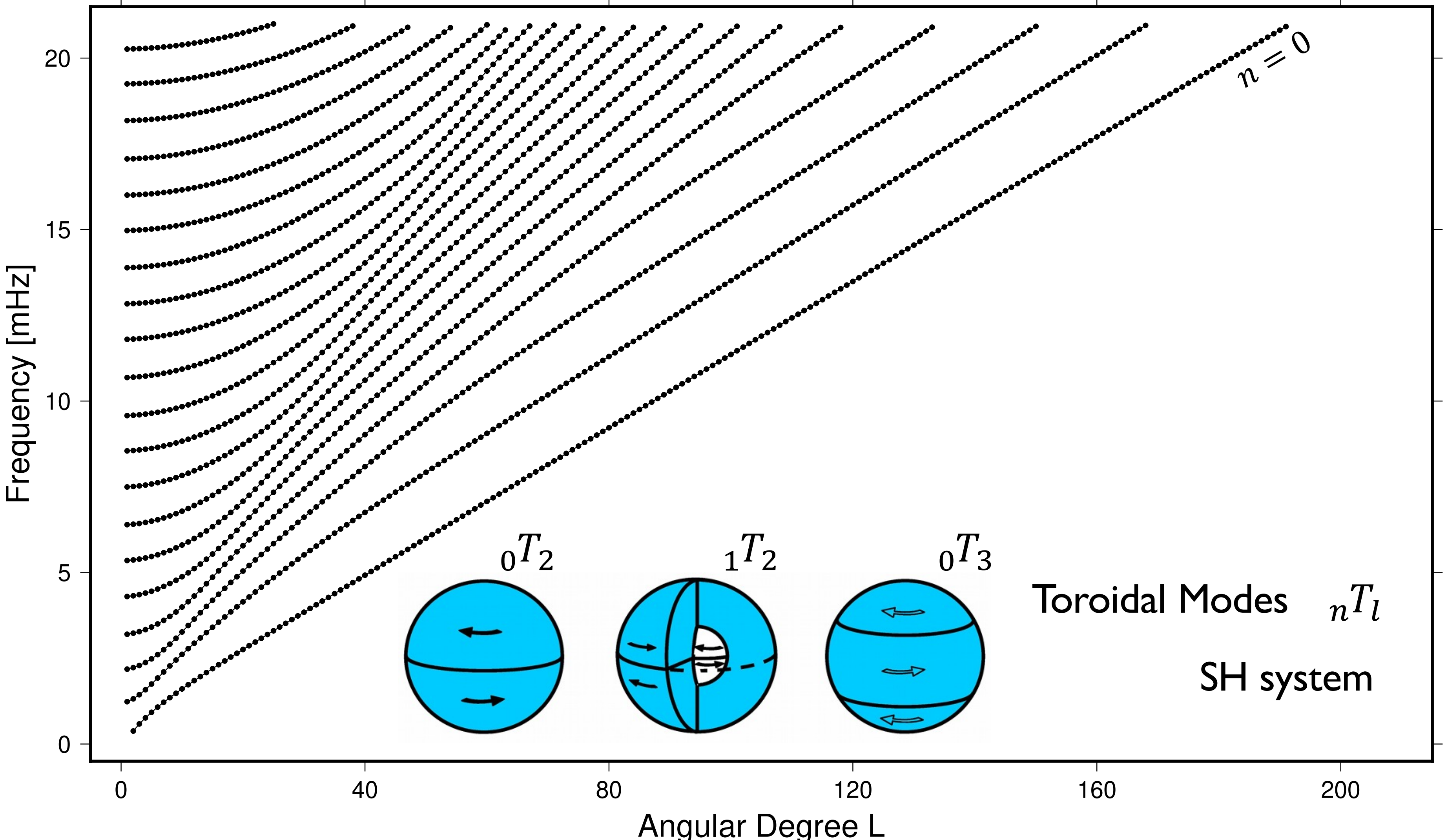


- ➢ Solve this eigenvalue problem (for toroidal modes)
- Choose l and a trial frequency ω
- Start from the Core-Mantle Boundary (CMB, fluid-solid interface, 2891 km depth)

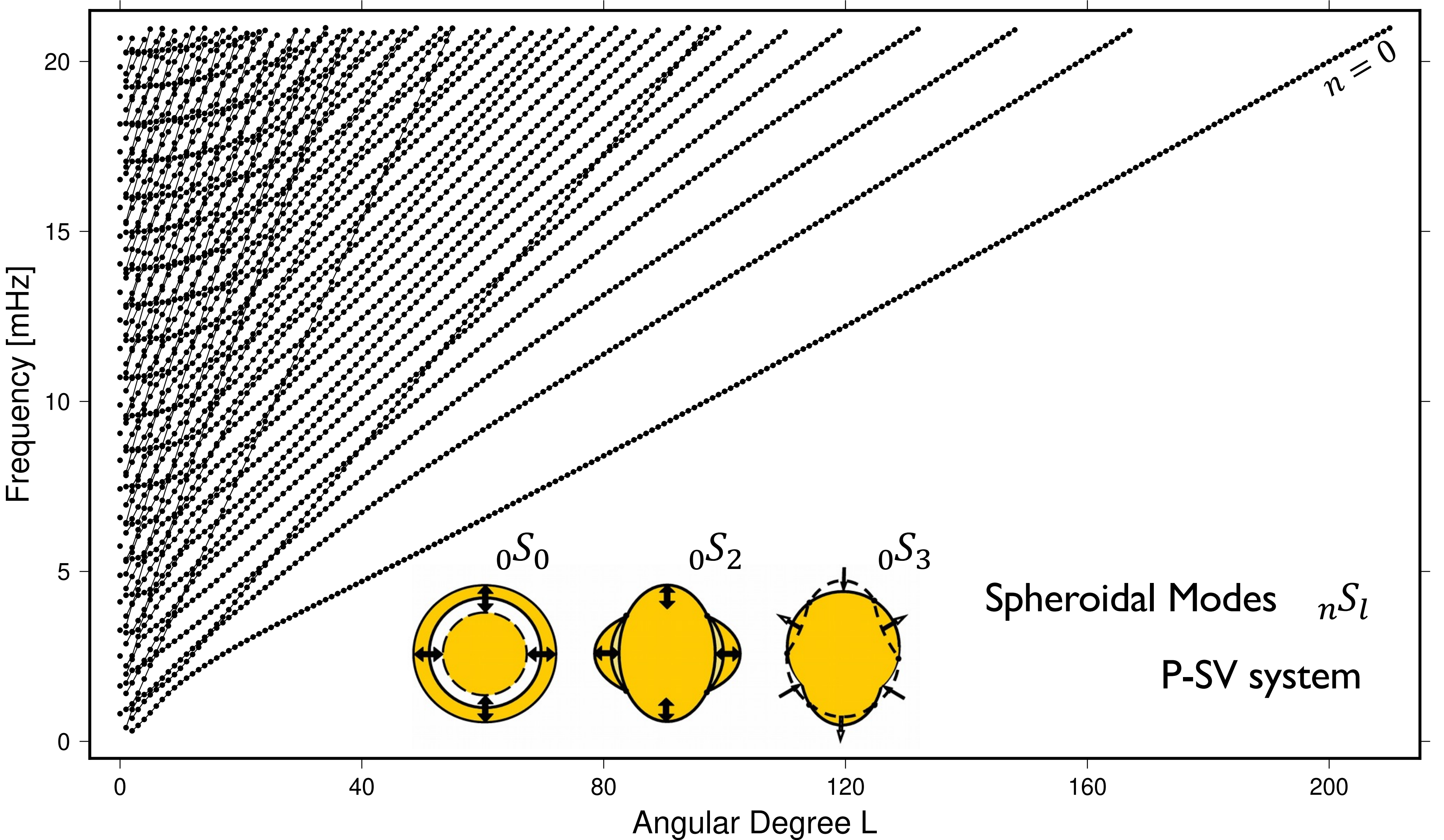
$$W(r = \text{CMB}) = \text{any nonzero} \rightarrow 1 \qquad\qquad T(r = \text{CMB}) = 0$$

- Integrate upward until reaching the free surface
- Satisfying the free-surface boundary condition gives constraint on mode frequency





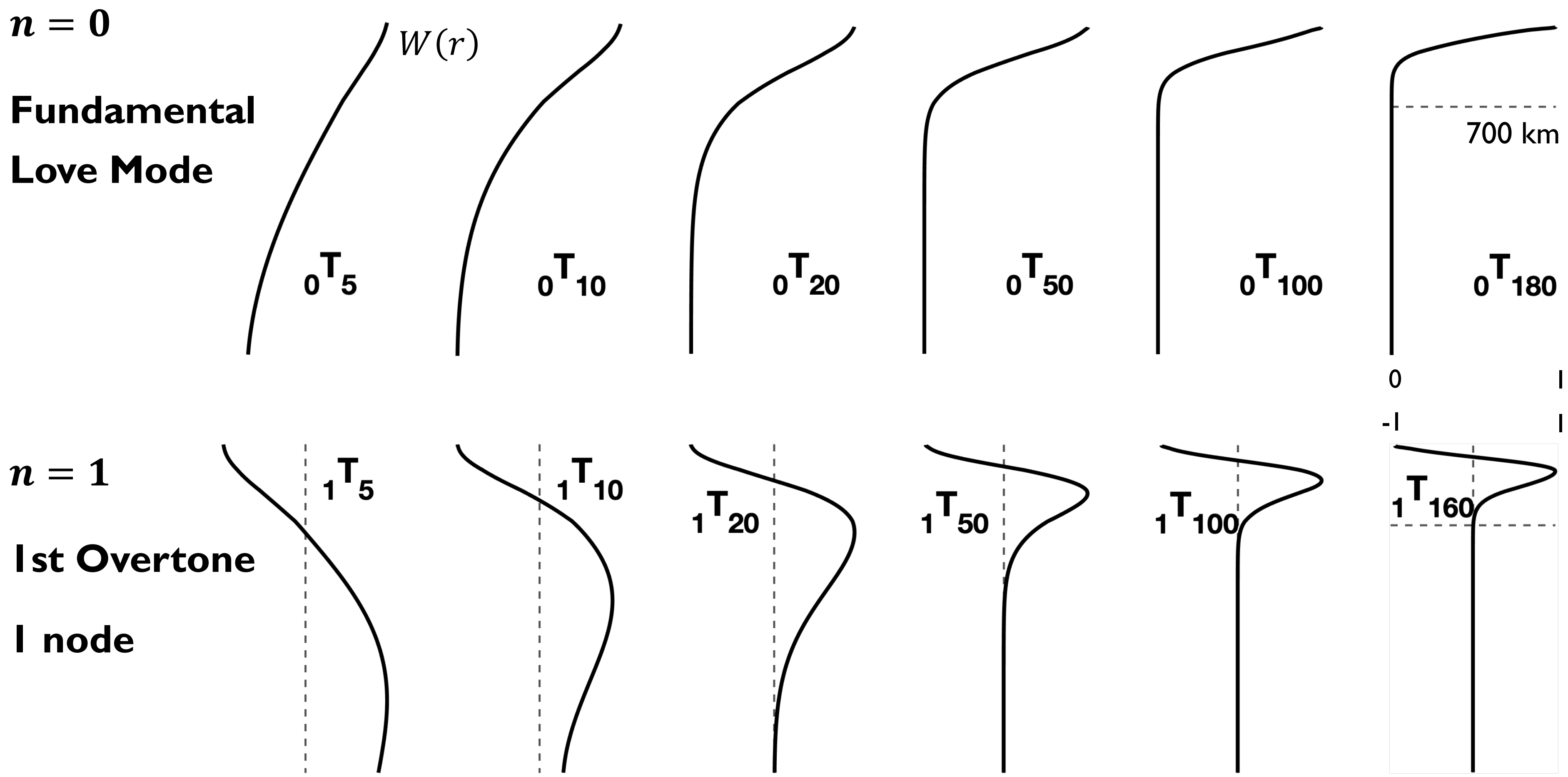
Frequency [mHz]
0
5
10
15
20
Angular Degree L
0
40
80
120
160
200
$n = 0$
$_0T_2$
$_1T_2$
$_0T_3$
Toroidal Modes $_nT_l$
SH system



Frequency [mHz]
Angular Degree L
0
5
10
15
20
0
40
80
120
160
200
$n = 0$
${}_0S_0$
${}_0S_2$
${}_0S_3$
Spheroidal Modes ${}_nS_l$
P-SV system

Radial Eigenfunction of Toroidal Modes

$_nW_l(r)\mathbf{C}_{lm}(\theta, \phi)$

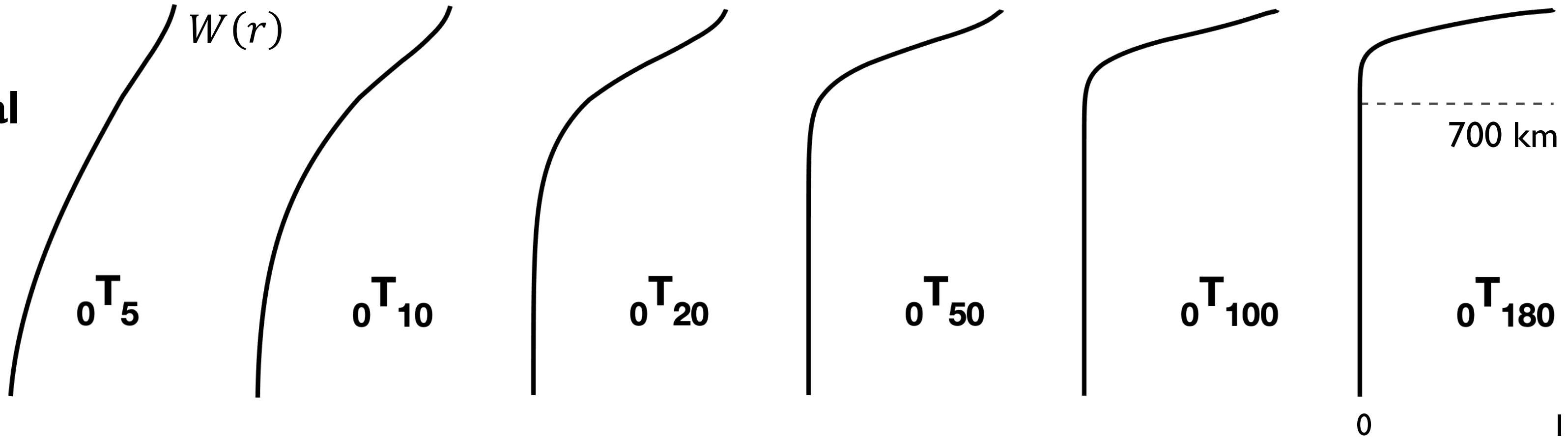


Excitation of surface waves

$_nW_l(r)\mathbf{C}_{lm}(\theta, \phi)$

$n = 0$

Fundamental Love Mode

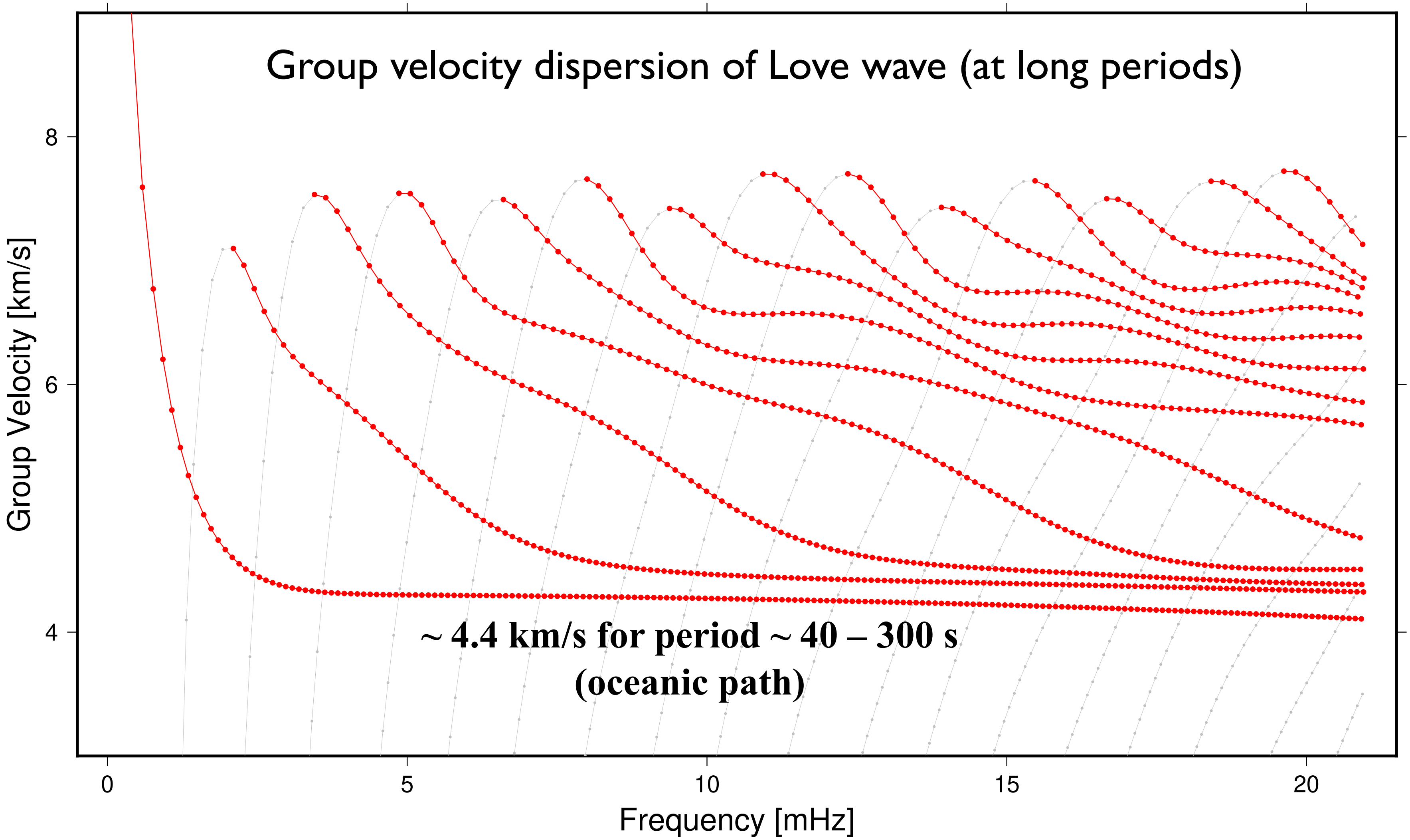


Excitation "coefficient": Radial eigenfunction evaluated at source depth

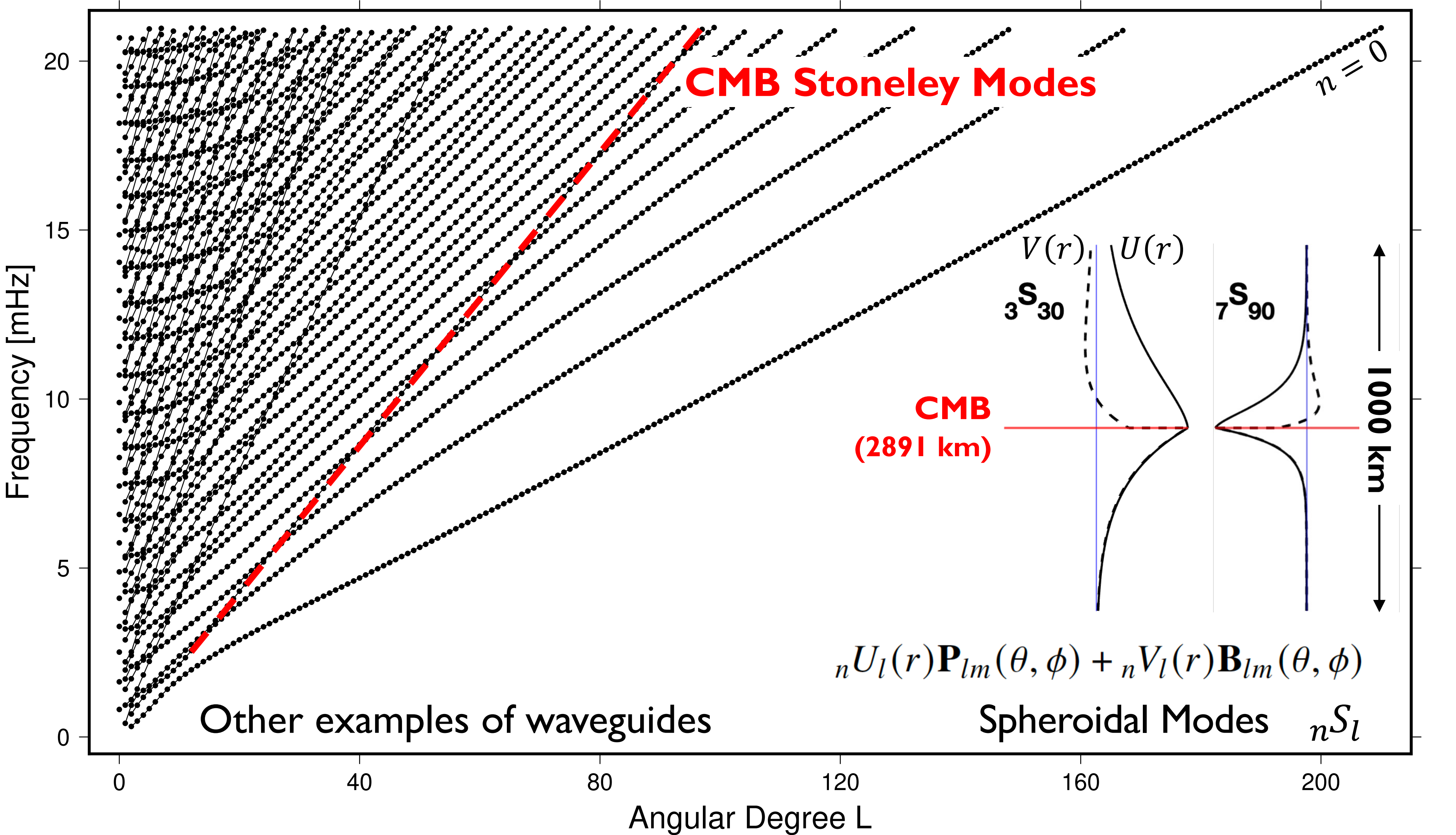
$$\mathbf{G}(\mathbf{r}, t; \mathbf{r}_0) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\sin({}_n\omega_l t)}{{}_n\omega_l} \, {}_n\mathbf{s}_{lm}(\mathbf{r}) \, \boxed{{}_n\mathbf{s}_{lm}(\mathbf{r}_0)}, \qquad t > 0.$$

Source with depth h generates surface waves with wavelength $\lambda \gtrsim h$

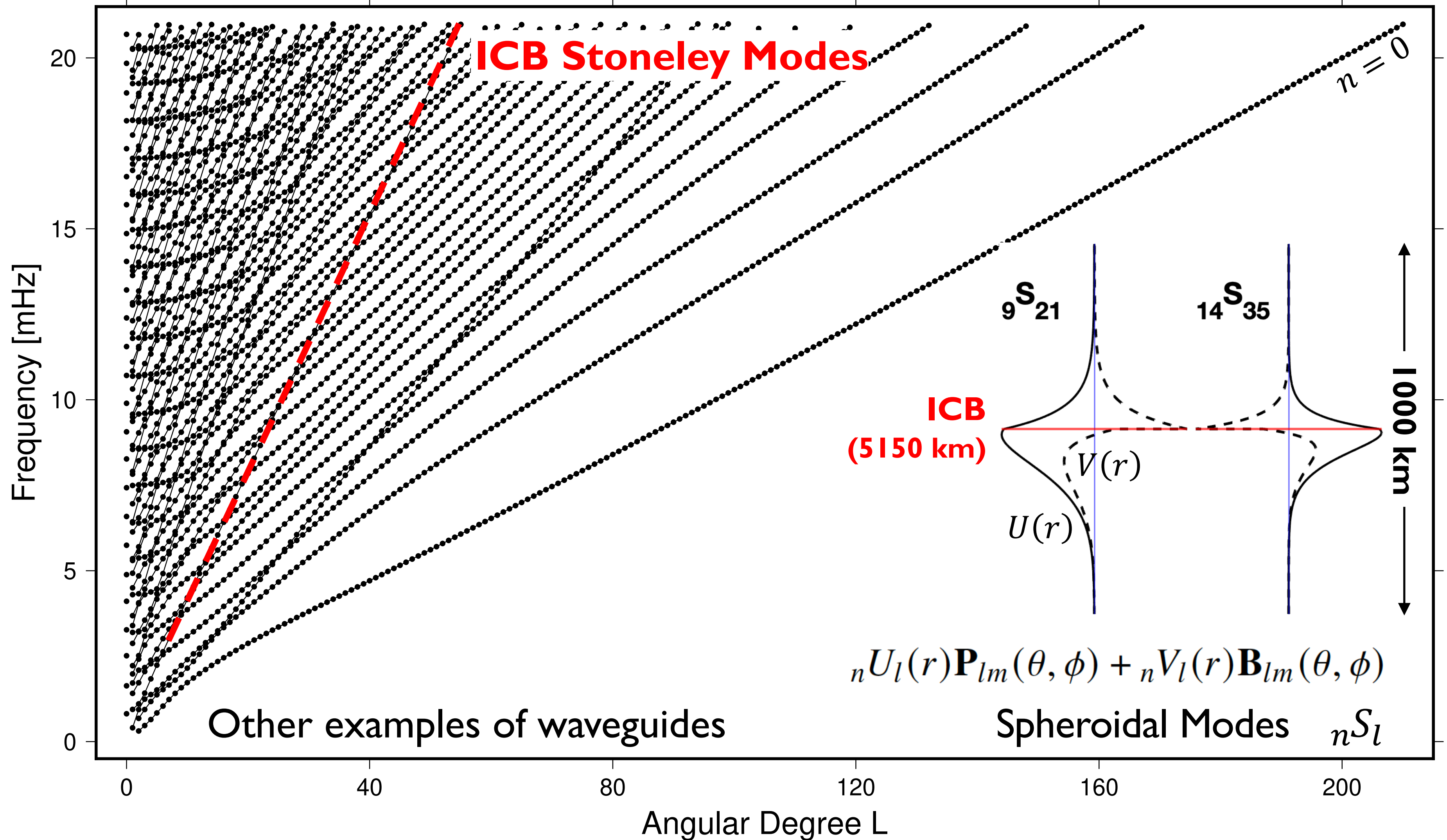
Simply speaking, shallow earthquakes can efficiently generate surface waves



Group velocity dispersion of Love wave (at long periods)
Group Velocity [km/s]
8
6
4
0
5
10
15
20
Frequency [mHz]
~ 4.4 km/s for period ~ 40 – 300 s
(oceanic path)



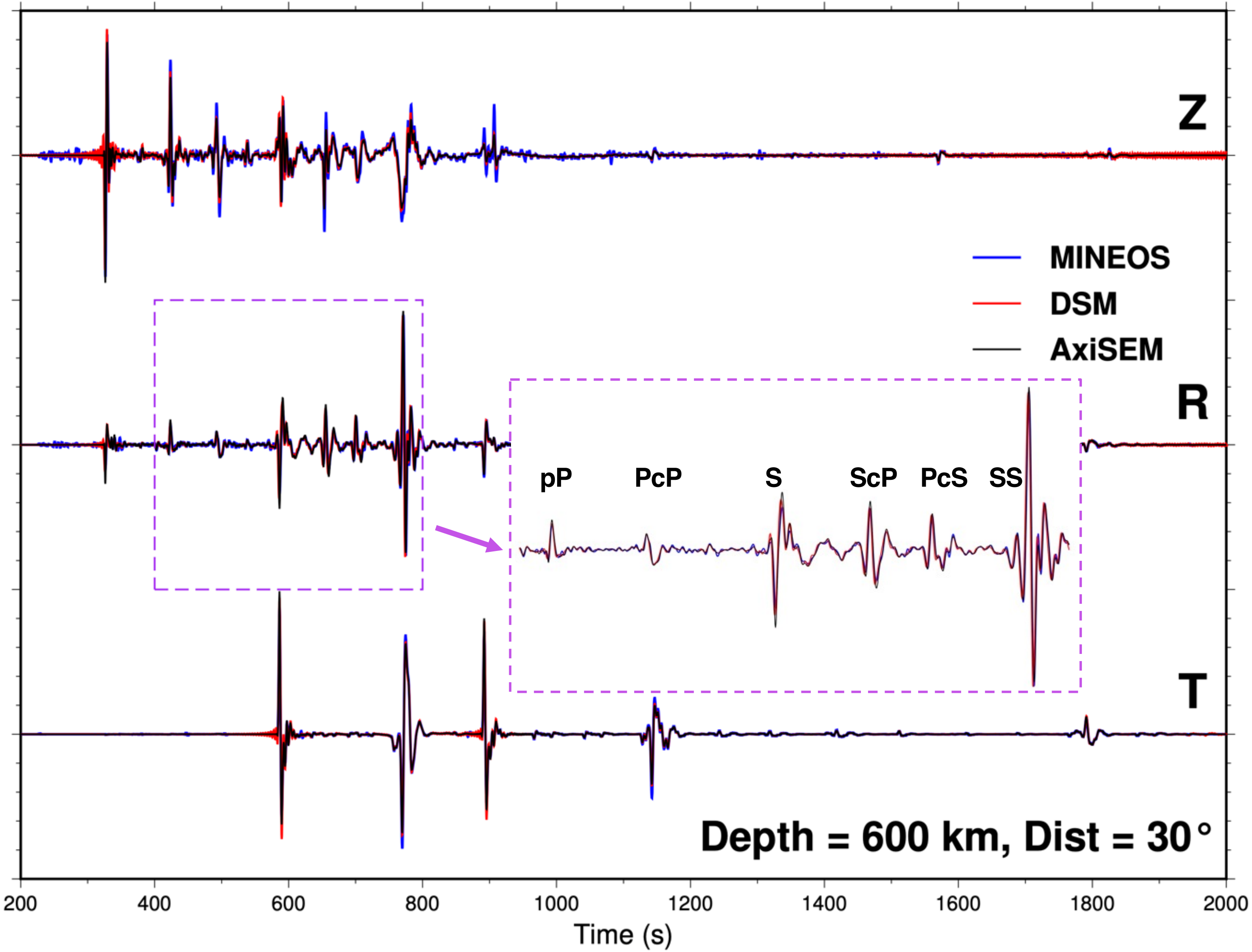
CMB Stoneley Modes
$n = 0$
$V(r)$
$U(r)$
$_3S_{30}$
$_7S_{90}$
CMB
(2891 km)
1000 km
$_nU_l(r)\mathbf{P}_{lm}(\theta, \phi) + {_nV_l}(r)\mathbf{B}_{lm}(\theta, \phi)$
Other examples of waveguides
Spheroidal Modes $_nS_l$
Frequency [mHz]
Angular Degree L
0
5
10
15
20
0
40
80
120
160
200


ICB Stoneley Modes
$n = 0$
$_9S_{21}$
$_{14}S_{35}$
ICB
(5150 km)
1000 km
$V(r)$
$U(r)$
$_nU_l(r)\mathbf{P}_{lm}(\theta, \phi) + {}_nV_l(r)\mathbf{B}_{lm}(\theta, \phi)$
Other examples of waveguides
Spheroidal Modes $_nS_l$
Frequency [mHz]
Angular Degree L
0
5
10
15
20
0
40
80
120
160
200

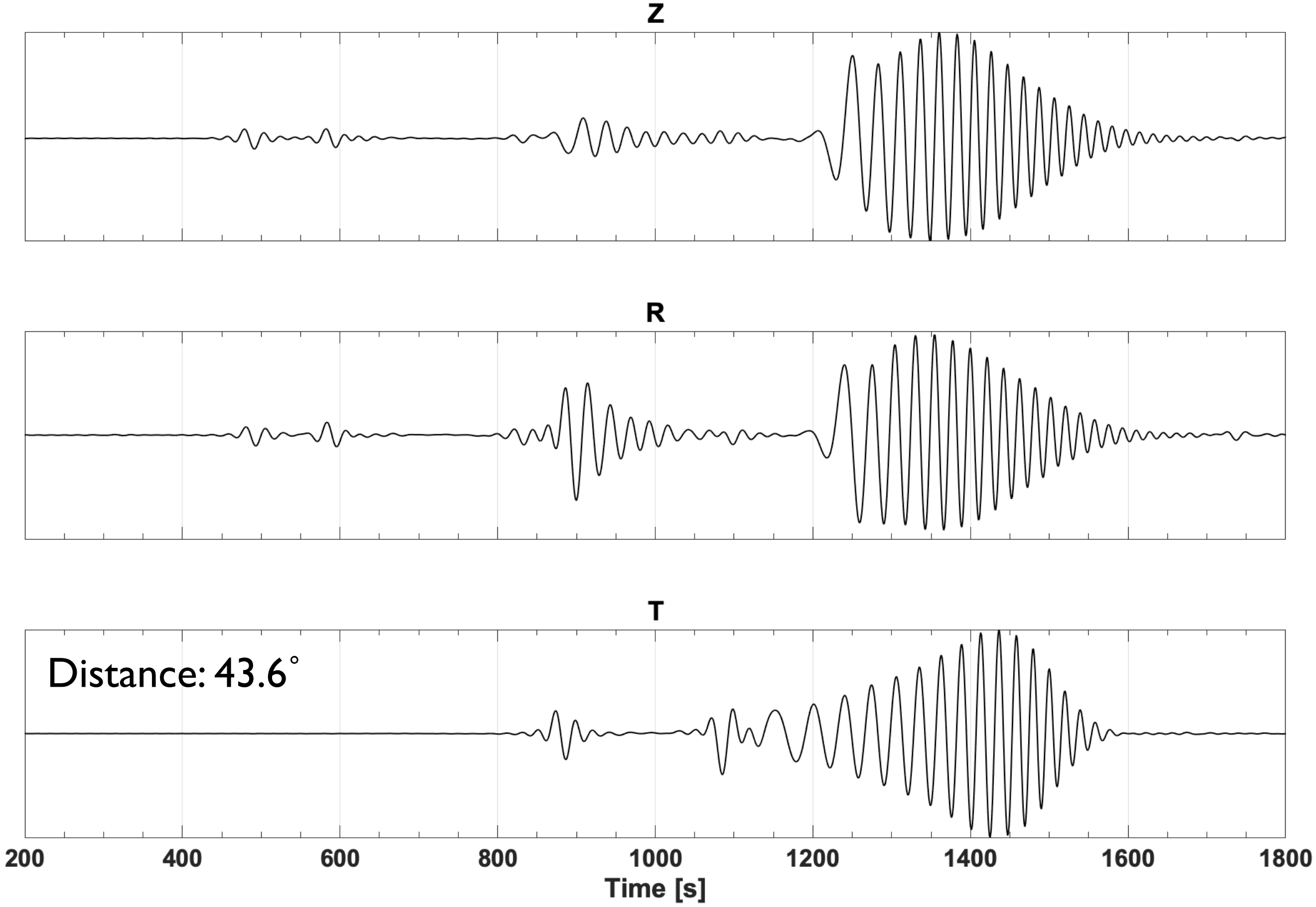
Calculate synthetic seismograms from mode summation

Deep earthquake (Depth 600 km)

Low-pass 0.2 Hz



Calculate synthetic seismograms from mode summation



Shallow earthquake (Depth 10 km)

Low-pass 0.05 Hz

In this lecture ……

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(Complete Basis!)

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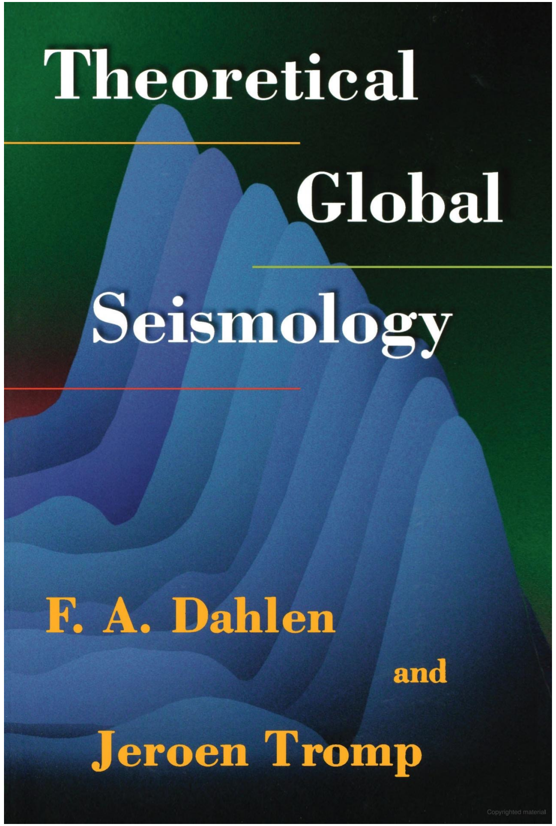
Recommended Reference (if you are interested!)



Chapter 8.7 & 8.8: Gallery of radial eigenfunctions

Chapter 10: Synthetic seismograms from mode summation

Singh, S.J., Rani, S. (2020). Free Oscillations of the Earth. In: Encyclopedia of Solid Earth Geophysics.

Thank You!