ESS 363F Geophysical Fluid Dynamics

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Topics to be covered:

1. Equations of motion in a rotating reference frame
    - Boussinesq approximation, shallow water equations, Coriolis force
2. Effects of stratification, inertia-gravity waves (IGWs)
    - Dispersion relations, polarization relation, energy flux
    - IGWs in inhomogeneous media, WKB approximation
3. Balanced motions
    - Geostrophic and gradient wind balance, inertial instability
4. Vorticity and potential vorticity
    - Vorticity equation, Taylor-Proudman effect, thermal wind balance
    - Potential vorticity in shallow water systems and continuously stratified media
5. Geostrophic theory
    - Geostrophic adjustment, Poincaré and Kelvin waves
    - Quasi-geostrophic equations for the shallow water system: Barotropic Rossby waves
    - Quasi-geostrophic equations for layered systems: Baroclinic Rossby waves
6. Baroclinic instability
7. Effects of friction
    - Ekman layer
    - Frictional spin-down of geostrophic flows
8. Two-dimensional turbulence

Textbooks:

- Geoffrey K. Vallis, *Atmospheric and Oceanic Fluid Dynamics, Fundamentals and Large-scale Circulation*
- Adrian E. Gill, *Atmosphere-Ocean Dynamics*

Introduction

➢ Flow dynamics under rotation and stratification

Earth's rotation

The Earth's rotation gives an angular velocity $\Omega = 7.27 \times 10^{-5}$ rad/s. Fluid motions that vary with time scales T longer than a day, or whose inherent spin is less than Ω are influenced by the Earth's rotation.

$$T \geq \frac{1}{\Omega}, \qquad U \leq \Omega L$$

Stratification

The stably layered ocean has a buoyancy frequency $N^2 = 10^{-4} \sim 10^{-6}$ (rad/s)^2. To move a fluid parcel upward with distance H, the potential energy increases as

$$\Delta E_P \sim \rho_o N^2 H \cdot H = \rho_o N^2 H^2$$

If in this vertical scale H, the flow kinetic energy is less than this amount, the fluid motion is influenced by stratification.

$$E_k \sim \rho_o U^2 < \Delta E_P, \qquad U < NH$$

Flows constrained by rotation and stratification

Using the (temporal) Rossby number and Froude number, we consider the following flows

$$\mathrm{Ro_t} = \frac{1}{\Omega T} < 1, \qquad \mathrm{Ro} = \frac{U}{\Omega L} < 1, \qquad \mathrm{Fr} = \frac{U}{NH} < 1$$

As an example, geophysical flows that meet these conditions include

- ♦ **Waves with slow temporal evolution**: Inertia-gravity, Kelvin, Rossby waves
- ♦ **Balanced motions**: Geostrophic and cyclostrophic currents and vortices
- ♦ **Large-scale instabilities**: Inertial, symmetric, barotropic, and baroclinic instabilities

➢ Characteristics of some geophysical fluid flows

Inertia-gravity waves: Phase and group velocities in different directions

Kelvin waves: Equatorial and coastal waves with monthly time scales

Rossby waves: Propagating to the west with yearly time scales

Geostrophic balance: Coriolis force maintains the strong cross-stream density difference

Two-dimensional turbulence: Inverse cascade, energy transferred from small to large scales

Governing Equations for Oceanic Flows

- Governing equations for an oceanic flow

Variables: u, v, w, p, ρ, T, S

Equations: EOS, momentum×3, continuity (mass), energy, salinity

Conservation equation

For a scalar quantity C and its corresponding flux $\boldsymbol{F}_c$, the conservation law is

$$\frac{\partial C}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}_c = 0, \qquad \int_V \frac{\partial C}{\partial t}\,\mathrm{d}V = -\oint_S \boldsymbol{F}_c \cdot \hat{\boldsymbol{n}}\,\mathrm{d}S = -\int_V \boldsymbol{\nabla} \cdot \boldsymbol{F}_c\,\mathrm{d}V$$

Conservation of mass (continuity equation)

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = 0$$

Scaling analysis shows that

$$\frac{|\mathrm{D}\rho/\mathrm{D}t|}{|\rho(\boldsymbol{\nabla} \cdot \boldsymbol{u})|} \sim \frac{\delta\rho/T}{\rho U/L} \sim \frac{\delta\rho}{\rho_o} \ll 1$$

Here we consider the time scale $T \sim L/U$. We thus neglect the density variation in time and consider incompressible fluid with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$.

Conservation of momentum

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{F} - \rho g \hat{\mathbf{k}}, \qquad \boldsymbol{F} = \mu \nabla^2 \boldsymbol{u}$$

Equation of state

$$\rho = \rho_o[1 - \alpha(T - T_o) + \beta(S - S_o)], \qquad T_o = 25\ ^\circ\mathrm{C}, \qquad S_o = 35\ \mathrm{g/kg}$$

The thermal expansion and haline contraction coefficients are

$$\alpha = 2.6 \times 10^{-4}\ ^\circ\mathrm{C}^{-1}, \qquad \beta = 7.4 \times 10^{-4}\ (\mathrm{g/kg})^{-1}$$

Conservation of heat and salinity

The fluxes include convective and diffusive components

$$\boldsymbol{F}_T = \rho_o c_V \boldsymbol{u} T - k_T \nabla T, \qquad \frac{\mathrm{D}T}{\mathrm{D}t} = \kappa_T \nabla^2 T, \qquad \kappa_T = 10^{-7}\ \mathrm{m^2/s}$$

$$\boldsymbol{F}_S = \boldsymbol{u} S - \kappa_S \nabla S, \qquad \frac{\mathrm{D}S}{\mathrm{D}t} = \kappa_S \nabla^2 S, \qquad \kappa_S = 10^{-9}\ \mathrm{m^2/s}$$

- Boussinesq approximation

Variation of density in the ocean

Denote $\rho = \rho_o + \rho'$ with reference value $\rho_o = 1024$ kg/m^3. Over depth, $\delta\rho < 40$ kg/m^3. At ocean surface, we have $\delta\rho < 5$ kg/m^3. The relative variation of sea water density is very small.

Horizontal momentum equations

Based on the following approximation

$$\frac{1}{\rho} = \frac{1}{\rho_o + \rho'} \approx \frac{1}{\rho_o}$$

The horizontal momentum equations are

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

The kinematic viscosity is $\nu = \mu/\rho_o = 10^{-6}$ m^2/s.

Vertical momentum equation

We first need the background hydrostatic balance

$$p = p_o + p', \qquad 0 = -\frac{1}{\rho_o}\frac{\partial p_o}{\partial z} - g$$

Keeping the first-order terms, the vertical momentum equation becomes

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} + \nu\nabla^2 w - \frac{\rho'}{\rho_o}g$$

Scaling relations

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad \Longrightarrow \qquad w \sim \frac{UH}{L}$$

$$\frac{\mathrm{D}u}{\mathrm{D}t} \sim \frac{1}{\rho_o}\frac{\partial p}{\partial x} \qquad \Longrightarrow \qquad \frac{1}{\rho_o}\frac{\partial p}{\partial z} \sim \frac{U}{T}\cdot\frac{L}{H}$$

Hydrostatic balance

From scaling analysis, we have

$$\frac{|\mathrm{D}w/\mathrm{D}t|}{|\rho_o^{-1}\partial p'/\partial z|} \sim \frac{UH/LT}{UL/TH} \sim \left(\frac{H}{L}\right)^2$$

Therefore, when the aspect ratio of the flow $H/L \ll 1$, we can also consider hydrostatic balance for the perturbed pressure

$$0 = -\frac{\partial p'}{\partial z} - \rho' g$$

- Summary of governing equations

Background state

$$\rho = \rho_o + \rho', \qquad p = p_o + p', \qquad \frac{\partial p_o}{\partial z} = -\rho_o g$$

Equation of state

$$\rho = \rho_o[1 - \alpha(T - T_o) + \beta(S - S_o)]$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum equations

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$
$$\frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$
$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} + \nu\nabla^2 w - \frac{\rho'}{\rho_o} g$$

Diffusion equations

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \kappa_T \nabla^2 T, \qquad \frac{\mathrm{D}S}{\mathrm{D}t} = \kappa_S \nabla^2 S$$

Equations of Motion in a Rotating Reference Frame

- Shallow water equations

Consider an inviscid fluid with free surface at $z = H + \eta$ and for shallow water $(H/L)^2 \ll 1$. The pressure is hydrostatic with $p = p_o$ satisfying

$$p = p_{\text{atm}} + \rho_o g(H + \eta - z), \qquad -\frac{1}{\rho_o}\nabla_h p = -g\nabla_h \eta$$

The pressure gradient force (PGF) is now written in surface height η. Horizontal momentum equations become

$$\frac{\mathrm{D}\boldsymbol{u}_h}{\mathrm{D}t} = -g\nabla_h \eta$$

If the velocity field is initially independent of depth, it will remain so throughout time since there are no terms in the equations of motion that will generate vertical structure.

- Uniformly rotating shallow water layer

The horizontal velocity field can be decomposed into

$$\boldsymbol{u}_h = u\widehat{\boldsymbol{\theta}} + v\hat{\mathbf{r}}, \qquad u = u' + \Omega r = r\frac{\mathrm{D}\theta}{\mathrm{D}t}$$

The unit vectors (always following the fluid particle) are given as

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}, \qquad \widehat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

Therefore, we have

$$\frac{\mathrm{D}\hat{\mathbf{r}}}{\mathrm{D}t} = \frac{u}{r}\widehat{\boldsymbol{\theta}}, \qquad \frac{\mathrm{D}\widehat{\boldsymbol{\theta}}}{\mathrm{D}t} = -\frac{u}{r}\hat{\mathbf{r}}$$

The acceleration now becomes

$$\frac{\mathrm{D}\boldsymbol{u}_h}{\mathrm{D}t} = \frac{\mathrm{D}u}{\mathrm{D}t}\widehat{\boldsymbol{\theta}} + u\frac{\mathrm{D}\widehat{\boldsymbol{\theta}}}{\mathrm{D}t} + \frac{\mathrm{D}v}{\mathrm{D}t}\hat{\mathbf{r}} + v\frac{\mathrm{D}\hat{\mathbf{r}}}{\mathrm{D}t} = \left(\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{uv}{r}\right)\widehat{\boldsymbol{\theta}} + \left(\frac{\mathrm{D}v}{\mathrm{D}t} - \frac{u^2}{r}\right)\hat{\mathbf{r}}$$

From the shallow water equation, we have

$$\frac{\mathrm{D}v}{\mathrm{D}t} - \frac{u^2}{r} = -g\frac{\partial\eta}{\partial r}, \qquad \frac{\mathrm{D}u}{\mathrm{D}t} + \frac{uv}{r} = -\frac{g}{r}\frac{\partial\eta}{\partial\theta}$$

Angular momentum

The second equation states the evolution of angular momentum. Note that

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{u}_h = ru\hat{\mathbf{k}}, \qquad \frac{\mathrm{D}\boldsymbol{L}}{\mathrm{D}t} = \boldsymbol{r}\times\boldsymbol{F}$$

For both sides, we have

$$\frac{\mathrm{D}\boldsymbol{L}}{\mathrm{D}t} = \left(u\frac{\mathrm{D}r}{\mathrm{D}t} + r\frac{\mathrm{D}u}{\mathrm{D}t}\right)\hat{\mathbf{k}} = r\left(\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{uv}{r}\right)\hat{\mathbf{k}}, \qquad \boldsymbol{r}\times\boldsymbol{F} = -g\boldsymbol{r}\times\nabla_h\eta = -g\frac{\partial\eta}{\partial\theta}\hat{\mathbf{k}}$$

Governing equations in a rotating frame

We want governing equations for u' and v. Using $u = u' + \Omega r$, we obtain

$$\frac{\mathrm{D}v}{\mathrm{D}t} - \frac{u^2}{r} = -g\frac{\partial \eta}{\partial r} \quad \Longrightarrow \quad \frac{\mathrm{D}v}{\mathrm{D}t} - \frac{u'^2}{r} = -g\frac{\partial \eta}{\partial r} + \Omega^2 r + 2\Omega u'$$

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{uv}{r} = -\frac{g}{r}\frac{\partial \eta}{\partial \theta} \quad \Longrightarrow \quad \frac{\mathrm{D}u'}{\mathrm{D}t} + \frac{u'v}{r} = -\frac{g}{r}\frac{\partial \eta}{\partial \theta} - 2\Omega v$$

The Coriolis force (CF) shows up in addition to centrifugal force (CFF). Now decompose η as

$$\eta = \bar{\eta} + \eta', \qquad \bar{\eta} = \eta_o + \frac{\Omega^2 r^2}{2g}, \qquad -g\frac{\partial \bar{\eta}}{\partial r} = -\Omega^2 r$$

The final equations in the rotating frame become

$$\frac{\mathrm{D}v}{\mathrm{D}t} - \frac{u'^2}{r} = -g\frac{\partial \eta'}{\partial r} + 2\Omega u', \qquad \frac{\mathrm{D}u'}{\mathrm{D}t} + \frac{u'v}{r} = -\frac{g}{r}\frac{\partial \eta'}{\partial \theta} - 2\Omega v$$

- Inertial oscillations

Consider $\eta' = 0$ and because r is the distance to Earth's rotation axis, we have

$$\frac{|u'^2/r|}{|2\Omega u'|} \sim \frac{U}{2\Omega r} \ll 1, \qquad \mathrm{Ro} \ll 1$$

This leads to the oscillation equation

$$\frac{\mathrm{D}v}{\mathrm{D}t} = 2\Omega u', \qquad \frac{\mathrm{D}u'}{\mathrm{D}t} = -2\Omega v \qquad \Longrightarrow \qquad \frac{\mathrm{D}^2 u'}{\mathrm{D}t^2} + 4\Omega^2 u' = 0$$

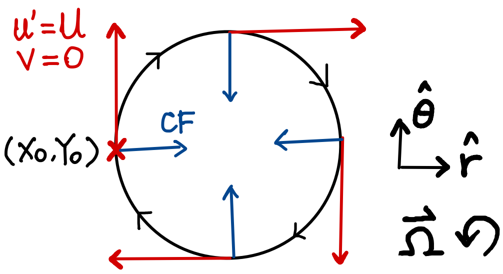
With initial conditions $u' = U$ and $v = 0$, we have

$$u' = U\cos(2\Omega t), \qquad v = U\sin(2\Omega \mathrm{t})$$

The inertial period is $T_i = \pi/\Omega$, and the inertial circle has a radius of $R_i = U/2\Omega$ **clockwise**.

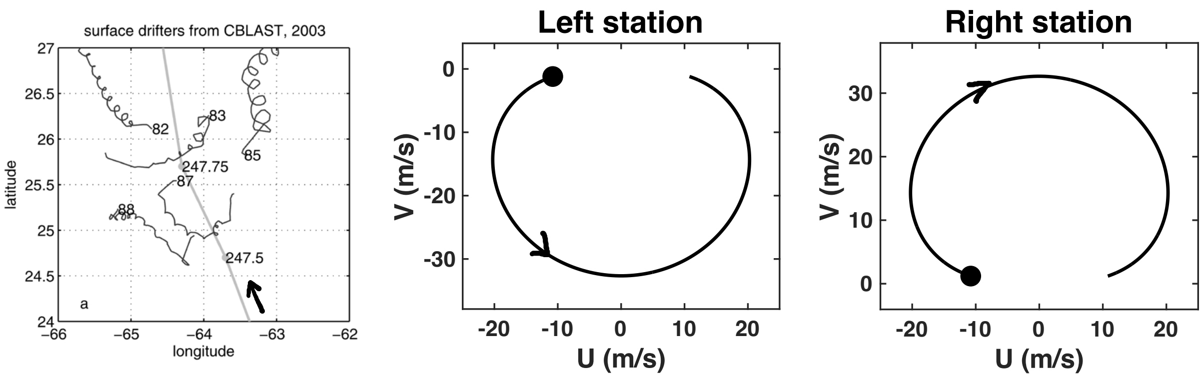
Interpretation of inertial circle

- In the rotating frame, Coriolis force (CF) serves as the centripetal force of the oscillation.
- In the inertial frame, the parcel oscillates radially due to the imbalance between PGF and CFF, in addition to the overall rotating motion.

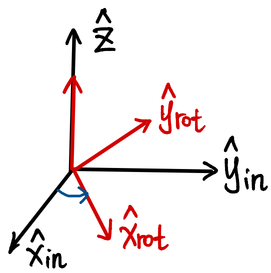


Inertial oscillations excited by hurricanes

It is observed that the cooling wakes and inertial oscillations are stronger to the right of the hurricane. For an observing point to the right of the storm, the wind evolves in the clockwise direction, possible to resonate with the inertial oscillation. However, for the point to the left, the wind evolves in the opposite direction.



- Transformation to a rotating frame



Denote the coordinates of inertial and rotating frames as $\hat{\mathbf{x}}_{\text{in}}$ and $\hat{\mathbf{x}}_{\text{rot}}$. The Earth's rotation gives the angular velocity $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$. The rotating frame shares the same z-axis. The time derivatives of the rotating unit vectors are

$$\frac{\mathrm{D}\hat{\mathbf{x}}_{\text{rot}}}{\mathrm{D}t} = \mathbf{\Omega} \times \hat{\mathbf{x}}_{\text{rot}}, \qquad \frac{\mathrm{D}\hat{\mathbf{y}}_{\text{rot}}}{\mathrm{D}t} = \mathbf{\Omega} \times \hat{\mathbf{y}}_{\text{rot}}$$

For an arbitrary vector $\mathbf{A}$, we have

$$\left(\frac{\mathrm{D}\mathbf{A}}{\mathrm{D}t}\right)_{\text{in}} = \left(\frac{\mathrm{D}\mathbf{A}}{\mathrm{D}t}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{A}$$

Therefore, the velocity and acceleration are given as

$$\boldsymbol{u}_{\text{in}} = \boldsymbol{u}_{\text{rot}} + \mathbf{\Omega} \times \boldsymbol{X}, \qquad \boldsymbol{a}_{\text{in}} = \boldsymbol{a}_{\text{rot}} + 2\mathbf{\Omega} \times \boldsymbol{u}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{\Omega} \times \boldsymbol{X}$$

The momentum equation in the rotating frame thus becomes

$$\rho(\boldsymbol{a}_{\text{rot}} + 2\mathbf{\Omega} \times \boldsymbol{u}_{\text{rot}} + \mathbf{\Omega} \times \mathbf{\Omega} \times \boldsymbol{X}) = -\nabla p + \mu\nabla^2\boldsymbol{u}_{\text{rot}} - \rho g\hat{\mathbf{k}}$$

- Geopotential

Both the gravitational force and the centrifugal force can be expressed by potentials,

$$\Phi = gz - \frac{\Omega^2 r^2}{2}, \qquad -\nabla\Phi = -g\hat{\boldsymbol{z}} - \mathbf{\Omega} \times \mathbf{\Omega} \times \boldsymbol{r}$$

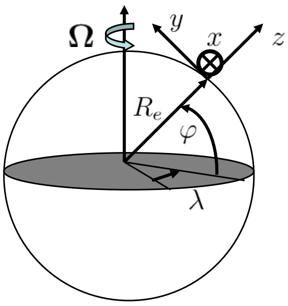
As an example, for a rotating water tank, the geopotential surfaces are paraboloids defined as

$$z - \frac{\Omega^2 r^2}{2g} = \text{const.}$$

The water surface $\bar{\eta}$ is also a geopotential surface in this case, since PGF balances with $\nabla\Phi$.

For the rotating planet, geopotential surfaces are oblate spheroids. It is convenient to redefine the horizontal coordinates to be parallel to geopotential surfaces, and then confines the effective gravitational force to the vertical momentum equation in the new coordinate system.

- Simplified equations of motion on a sphere



At location (lat, lon) $= (\varphi_o, \lambda_o)$, define a local Cartesian coordinate on a plane tangent to the sphere.

$$x = R_e \cos\varphi \cdot (\lambda - \lambda_o), \qquad y = R_e(\varphi - \varphi_o), \qquad z = r - R_e$$

The **zonal** direction is x, and the **meridional** direction is y.

The Coriolis parameter is obtained from the projection of $2\boldsymbol{\Omega}$

$$\boldsymbol{f} = 2\boldsymbol{\Omega} = 2\Omega\sin\varphi\,\hat{\mathbf{k}} + 2\Omega\cos\varphi\,\hat{\mathbf{j}} = f\hat{\mathbf{k}} + f_*\hat{\mathbf{j}}$$

Momentum equation

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\boldsymbol{u} - \rho g\hat{\boldsymbol{k}}$$

In the rotating frame, the momentum equation under Boussinesq assumption becomes

$$\frac{\mathrm{D}\boldsymbol{u}_{\mathrm{rot}}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u}_{\mathrm{rot}} = -\frac{1}{\rho_o}\nabla p' + \nu\nabla^2\boldsymbol{u}_{\mathrm{rot}} - \frac{\rho'}{\rho_o}g_{\mathrm{eff}}\,\hat{\mathbf{k}}$$

The subscripts and primes will be neglected from now on.

Coriolis parameter

$$\boldsymbol{f}\times\boldsymbol{u} = (f_*w - fv)\hat{\mathbf{i}} + fu\hat{\mathbf{j}} - f_*u\hat{\mathbf{k}}$$

Based on the following approximation

$$\varphi - \varphi_o = \frac{y}{R_e} \sim \frac{L}{R_e} \ll 1$$

The Taylor expansion of Coriolis parameter gives

$$f = 2\Omega\sin\varphi \approx 2\Omega\sin\varphi_o + 2\Omega\cos\varphi_o\cdot(\varphi - \varphi_o)$$

The beta-plane approximation assumes

$$f = f_o + \beta y, \qquad \beta = \frac{2\Omega}{R_e}\cos\varphi_o$$

The f-plane approximation assumes

$$f = f_o, \qquad \frac{\beta L}{f_o} \ll 1$$

The traditional approximation neglects f_* and assumes

$$(\boldsymbol{f}\times\boldsymbol{u})_x \approx -fv, \qquad \frac{f_*W}{fU} \sim \frac{H}{L}\cot\varphi \ll 1$$

Inertia-Gravity Wave (IGW)

- Static stability in a stratified fluid

Consider the fluid density profile

$$\rho(z) = \rho_o + \bar{\rho}(z), \qquad 0 = -\frac{1}{\rho_o}\frac{\mathrm{d}\bar{p}}{\mathrm{d}z} - \frac{\bar{\rho}}{\rho_o}g$$

If a fluid parcel is displaced adiabatically from $z = z_o$ to $z = z_o + h$, its density is preserved as $\bar{\rho}_p = \bar{\rho}(z_o)$. The vertical momentum equation, evaluated at parcel location $z = z_o + h$, is

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_o}\frac{\mathrm{d}\bar{p}}{\mathrm{d}z} - \frac{\bar{\rho}_p}{\rho_o}g = -\frac{g}{\rho_o}\left[\bar{\rho}_p - \bar{\rho}(z_o + h)\right]$$

For small displacement, we have

$$\bar{\rho}(z_o + h) \approx \bar{\rho}(z_o) + \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}(z_o)\, h$$

Now we notice the buoyancy frequency N, and obtain the oscillation equation

$$N^2 = -\frac{g}{\rho_o}\frac{\partial\bar{\rho}}{\partial z}, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = \frac{\mathrm{D}^2 h}{\mathrm{D}t^2} \approx -N^2 h$$

Stable stratification means $N^2 > 0$ with fluid parcels oscillating around equilibrium positions at buoyancy frequency N. Unstable stratification implies convection, fluid parcels accelerating away from equilibrium positions.

Energetics

Under stable stratification, an external force is required to move a fluid parcel away from its original equilibrium position

$$F_{ext} = -F_{buoy} = N^2 h, \qquad E = \int_0^H F_{ext}\,\mathrm{d}h = \frac{1}{2}N^2 H^2$$

Therefore, fluid parcels tend to move along the nearly horizontal density surfaces.

Inertia-gravity oscillation

Consider a fluid parcel moves in the yz-plane with angle α from y-axis. In the slanted direction, the small displacement is δ and the restoring force is F_δ.

$$\boldsymbol{\delta} = h\hat{\mathbf{k}} + Y\hat{\mathbf{j}} = \delta\sin\alpha\,\hat{\mathbf{k}} + \delta\cos\alpha\,\hat{\mathbf{j}}, \qquad \boldsymbol{F} = F_z\hat{\mathbf{k}} + F_y\hat{\mathbf{j}} = -N^2 h\hat{\mathbf{k}} - fu\hat{\mathbf{j}}$$

In x-direction, we obtain the conservation of absolute momentum M

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = \frac{\mathrm{D}}{\mathrm{D}t}(u - fY) = 0, \qquad M = u - fY = \text{const.}$$

With the initial conditions $u = 0$ and $Y = 0$, we can set $u = fY$. This gives

$$\boldsymbol{F} = -N^2 h\hat{\mathbf{k}} - f^2 Y\hat{\mathbf{j}}, \qquad F_\delta = \boldsymbol{F}\cdot\boldsymbol{\delta} = -\delta^2(N^2\sin^2\alpha + f^2\cos^2\alpha)$$

The angular frequency of the oscillation is

$$\omega^2 = N^2 \sin^2 \alpha + f^2 \cos^2 \alpha$$

For ocean we have $N > f$. In the vertical direction $\omega_{\max} = N$ governed by buoyancy. In the horizontal direction $\omega_{\min} = f$, which is the inertial oscillation.

- Inertia-gravity wave (IGW)

With the following assumptions:

a. Small perturbation caused by waves such that advection terms are neglected
b. Constant buoyancy frequency from the background profile

$$\rho = \rho_o + \bar{\rho}(z) + \rho'(\boldsymbol{x}, t), \qquad p = p_o + \bar{p} + p', \qquad N^2 = -\frac{g}{\rho_o}\frac{\partial \bar{\rho}}{\partial z}, \qquad b = -\frac{\rho'}{\rho_o} g$$

c. Neglect N-S component of the Coriolis parameter, i.e., f-plane approximation
d. Inviscid and adiabatic fluid: $\nu = 0$ and $\mathrm{D}\rho/\mathrm{D}t = 0$

From the conservation of density, neglecting second order terms gives

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho \approx \frac{\partial \rho'}{\partial t} + w \frac{\partial \bar{\rho}}{\partial z} = 0$$

This leads to the buoyancy equation, and together with continuity

$$\frac{\partial b}{\partial t} = -N^2 w, \qquad \nabla \cdot \boldsymbol{u} = 0$$

the momentum equations become

$$\frac{\partial u}{\partial t} - f v = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x}, \qquad \frac{\partial v}{\partial t} + f u = -\frac{1}{\rho_o}\frac{\partial p'}{\partial y}, \qquad \frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} + b$$

Apply $-\nabla_h \cdot$ to the horizontal momentum equation, we have

$$\frac{\partial}{\partial t}(-\nabla_h \cdot \boldsymbol{u}_h) + f\zeta = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right) + f\zeta = \frac{1}{\rho_o}\nabla_h^2 p', \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Apply ∂_x to v-component and ∂_y to u-component, then subtract the two equations, we have

$$\frac{\partial \zeta}{\partial t} = -f(\nabla_h \cdot \boldsymbol{u}_h) = f\frac{\partial w}{\partial z}$$

Using this expression, we obtain

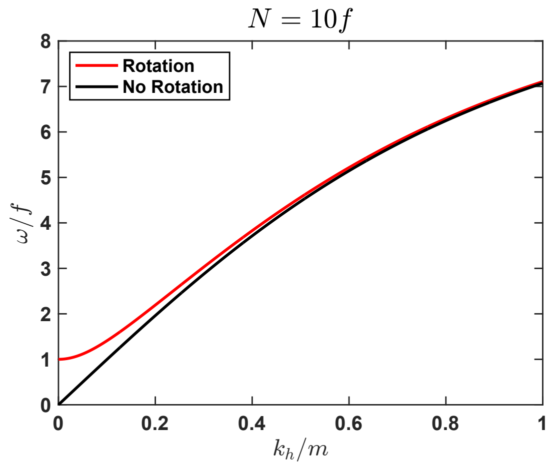
$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial w}{\partial z} = \frac{1}{\rho_o}\nabla_h^2 \frac{\partial p'}{\partial t}$$

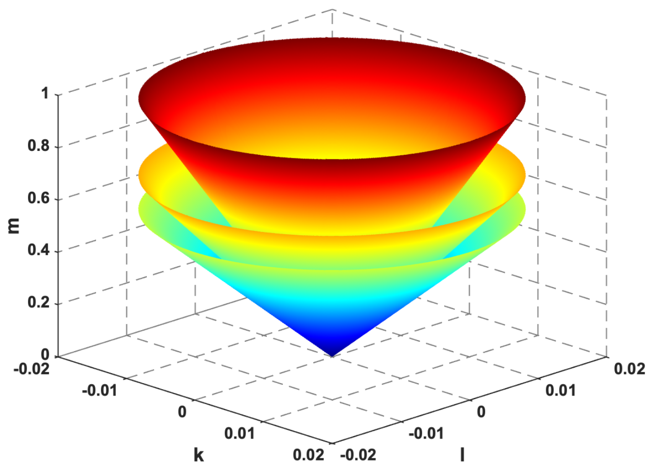
Apply ∂_z and use the following result

$$\frac{1}{\rho_o}\frac{\partial^2 p'}{\partial t \partial z} = -\frac{\partial^2 w}{\partial t^2} + \frac{\partial b}{\partial t} = -\frac{\partial^2 w}{\partial t^2} - N^2 w$$

The IGW equation is

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + f^2 \frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$





Dispersion relation

We seek a propagating wave solution $w = w_o \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]$. Denote the wavenumber as

$$\boldsymbol{k} = (k, l, m), \qquad |\boldsymbol{k}_h| = |\boldsymbol{k}| \cos\phi\,, \qquad m = |\boldsymbol{k}| \sin\phi$$

The dispersion relation of IGW becomes

$$\omega^2 |\boldsymbol{k}|^2 - f^2 m^2 - N^2 |\boldsymbol{k}_h|^2 = 0, \qquad \omega^2 = N^2 \cos^2\phi + f^2 \sin^2\phi$$

It can be also written as

$$\frac{m^2}{|\boldsymbol{k}_h|^2} = \frac{N^2 - \omega^2}{\omega^2 - f^2}$$

This shows that at a given frequency, the ratio of the vertical to the horizontal wavenumber is fixed. Note that for the fluid parcel motion, we previously obtain

$$\omega^2 = N^2 \sin^2\alpha + f^2 \cos^2\alpha$$

This shows that $\boldsymbol{k}$ and $\boldsymbol{u}$ are orthogonal to each other, which is also shown by continuity as

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Longrightarrow \qquad \boldsymbol{k} \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{k} \perp \boldsymbol{u}$$

The wavenumber is perpendicular to the particle motion. We see that $\omega \in [f, N]$ in a rotating fluid, and without rotation the frequency can go to zero. As $|\boldsymbol{k}_h| \ll m$, the parcel displacements are nearly horizontal, and gravity provides only a weak restoring force. The limiting case is the inertial oscillation.

When the forcing frequency ω falls out of the range $[f, N]$, the wave is **evanescent** around the source which does not freely propagate. As an example, the diurnal tides moving around the seamount can become sub-inertial for certain latitudes.

Phase and group velocities

$$\boldsymbol{c}_p = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\Phi} = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k}, \qquad \boldsymbol{c}_g = \nabla_{\boldsymbol{k}}\omega = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right)$$

From the dispersion relation, we have

$$c_g^x = \frac{N^2 - f^2}{\omega|\boldsymbol{k}|^4} km^2, \qquad c_g^y = \frac{N^2 - f^2}{\omega|\boldsymbol{k}|^4} lm^2, \qquad c_g^z = -\frac{N^2 - f^2}{\omega|\boldsymbol{k}|^4} m|\boldsymbol{k}_h|^2$$

The magnitude of the group velocity is thus

$$|\boldsymbol{c}_g| = \frac{N^2 - f^2}{\omega|\boldsymbol{k}|^3} m|\boldsymbol{k}_h|$$

For IGWs, the group velocity is perpendicular to the phase velocity, as $\boldsymbol{c}_g \cdot \boldsymbol{k} = 0$.

Energy propagation & Group velocity

For a narrow band signal propagating in x-direction with $\Delta k \ll k_o$

$$u(x,t) = \int_{k_o-\Delta k}^{k_o+\Delta k} \hat{A}(k - k_o)\, e^{i[kx-\omega(k)t]}\, \mathrm{d}k$$

Expand $\omega(k)$ around k_o gives

$$u(x,t) \approx e^{i[k_o x - \omega(k_o)t]} \int_{-\Delta k}^{\Delta k} \hat{A}(k)\, e^{ik[x-c_g(k_o)t]}\, \mathrm{d}k = A(x - c_g t) e^{i[k_o x - \omega(k_o)t]}$$

The energy of the wave propagates with the group velocity.

Polarization relations

Given the amplitude of the pressure wave, the velocity components have amplitudes

$$u_o = \frac{k\omega + ilf}{\omega^2 - f^2} \cdot \frac{p_o}{\rho_o}, \qquad v_o = \frac{l\omega - ikf}{\omega^2 - f^2} \cdot \frac{p_o}{\rho_o}, \qquad w_o = \frac{-m\omega}{N^2 - \omega^2} \cdot \frac{p_o}{\rho_o}$$

The buoyancy amplitude is

$$b_o = \frac{imN^2}{N^2 - \omega^2} \cdot \frac{p_o}{\rho_o}$$

The phase-averaged energy flux, Reynold stress is defined as

$$(\overline{p\boldsymbol{u}})_\Phi = \frac{1}{2\pi} \int_{\phi_0-\pi}^{\phi_o+\pi} p\boldsymbol{u}\, \mathrm{d}\phi\,, \qquad \left(\overline{u_i u_j}\right)_\Phi = \frac{1}{2\pi} \int_{\phi_0-\pi}^{\phi_o+\pi} u_i u_j\, \mathrm{d}\phi$$

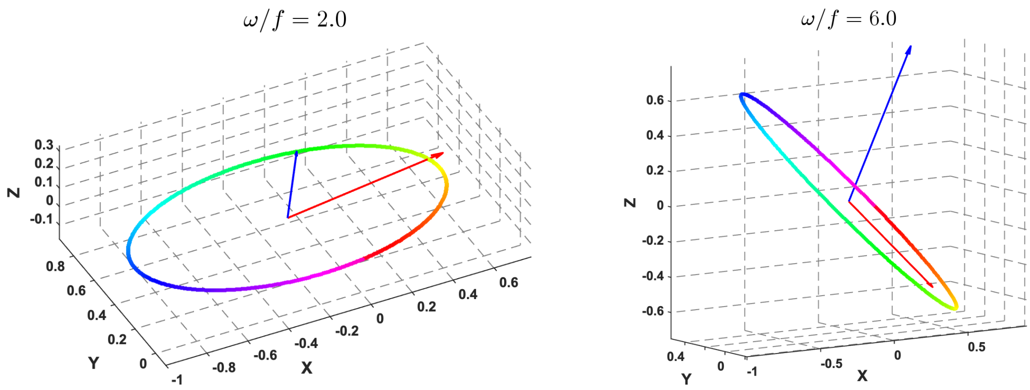
As an example, for a real amplitude p_o with $l = 0$

$$u_o = \frac{k\omega}{\omega^2 - f^2} \cdot \frac{p_o}{\rho_o}, \qquad u = u_o \cos\Phi\,, \qquad p = p_o \cos\Phi$$

The wave energy flux is

$$(\overline{pu})_\Phi = \frac{u_o p_o}{2} = \frac{1}{2} \cdot \frac{k\omega}{\omega^2 - f^2} \cdot \frac{p_o^2}{\rho_o}$$

Note that when $\omega \to f$, the inertial oscillation does not involve pressure perturbation, so we should use other amplitudes instead of p_o.



The polarization relations give the particle motion, which is a tilted ellipse. The group velocity points within the plane of particle motion.

- Near inertial wave ($\omega \to f$), more horizontal motion and energy propagation.
- Near buoyancy wave ($\omega \to N$), more vertical motion and energy propagation.

Consider a pressure wave moving in $\boldsymbol{k}$ direction in the horizontal plane

$$\frac{\partial u_{\parallel}}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_{\parallel}} + f u_{\perp}, \qquad \frac{\partial u_{\perp}}{\partial t} = -f u_{\parallel}$$

The plane wave solution gives

$$-i\omega\tilde{u}_{\perp} = -f\tilde{u}_{\parallel}, \qquad \left|\frac{\tilde{u}_{\perp}}{\tilde{u}_{\parallel}}\right| = \frac{f}{\omega} < 1$$

This implies that measurement of horizontal velocity components can estimate the frequency.

Kinetic and potential energy of IGW

The kinetic energy and potential energy are

$$\mathrm{KE} = \frac{\rho_o}{2} u_i u_i, \qquad \mathrm{PE} = \frac{\rho_o}{2} N^2 h^2 = \frac{\rho_o}{2}\frac{b^2}{N^2}$$

To see how potential energy relates to buoyancy, integrate the buoyancy equation

$$\int_0^t \frac{\partial b}{\partial t}\,\mathrm{d}t = -\int_0^t w N^2\,\mathrm{d}t\,, \qquad b(t) = -N^2 h(t)$$

The phase-averaged **energy density** satisfies

$$\overline{\mathrm{E}} = \overline{\mathrm{KE}} + \overline{\mathrm{PE}}, \qquad \frac{\overline{\mathrm{KE}}}{\overline{\mathrm{PE}}} = \frac{(\omega^2 + f^2)N^2 - 2\omega^2 f^2}{(\omega^2 - f^2)N^2}$$

- Near inertial waves with $\omega \to f$, $\overline{\mathrm{PE}} \to 0$ and energy is in the form of kinetic energy.
- Near buoyancy waves with $\omega \to N$, the energy is equipartitioned with $\overline{\mathrm{KE}} = \overline{\mathrm{PE}}$.

Energy equation

From the momentum equation, multiply by velocity component gives

$$u\frac{\partial u}{\partial t} = fuv - \frac{u}{\rho_o}\frac{\partial p}{\partial x}, \qquad v\frac{\partial v}{\partial t} = -fuv - \frac{v}{\rho_o}\frac{\partial p}{\partial y}, \qquad w\frac{\partial w}{\partial t} = -\frac{w}{\rho_o}\frac{\partial p}{\partial z} + wb$$

The kinetic energy is then governed by (using continuity $\nabla \cdot \boldsymbol{u} = 0$)

$$\frac{\partial \text{KE}}{\partial t} = -(\boldsymbol{u} \cdot \nabla p) + \rho_o wb = -(\nabla \cdot p\boldsymbol{u}) + \rho_o wb$$

The kinetic energy density is

$$\frac{\partial \overline{\text{KE}}}{\partial t} = -\nabla \cdot \boldsymbol{F}_{\text{E}} + \rho_o \overline{wb}, \qquad \boldsymbol{F}_{\text{E}} = \overline{p\boldsymbol{u}}$$

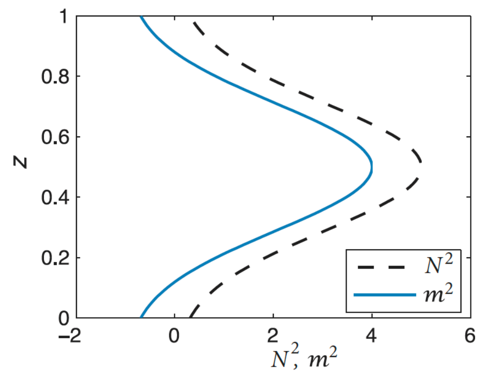
For the buoyancy equation, multiply by $\rho_o b/N^2$ gives

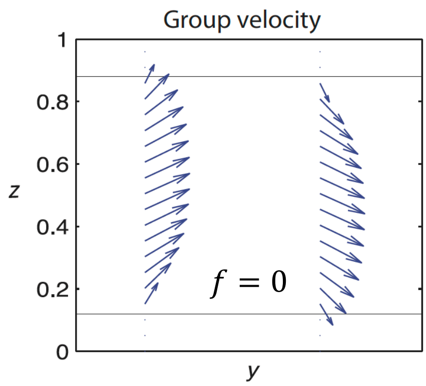
$$\frac{\rho_o b}{N^2}\frac{\partial b}{\partial t} = -\rho_o wb, \qquad \frac{\partial \overline{\text{PE}}}{\partial t} = -\rho_o \overline{wb}$$

From the polarization relations, the energy flux is proportional to the group velocity

$$\frac{\partial \overline{\text{E}}}{\partial t} = -\nabla \cdot \boldsymbol{F}_{\text{E}}, \qquad \boldsymbol{F}_{\text{E}} = \overline{p\boldsymbol{u}} = \overline{\text{E}}\boldsymbol{c}_g$$

- IGW in a non-uniform medium





Consider a layer of high stratification with larger $N^2(z)$. From dispersion relation, with fixed ω the particle motion angle α becomes smaller, the ray path becomes more horizontal. This is also shown by the group velocity direction (s is the horizontal distance)

$$\frac{\text{d}z}{\text{d}s} = \frac{c_g^z}{c_g^h} = -\frac{|\boldsymbol{k}_h|}{m} = -\text{sgn}(m)\sqrt{\frac{\omega^2 - f^2}{N^2 - \omega^2}}$$

WKB approximation

For a vertically varying $N^2(z)$, we seek $w = G(z)\, e^{i(kx+ly-\omega t)}$. The IGW equation becomes

$$G''(z) + m^2 G(z) = 0, \qquad m^2(z) = \frac{N^2(z) - \omega^2}{\omega^2 - f^2}(k^2 + l^2)$$

Consider $G(z) = A(z)\, e^{i\Phi(z)}$, the real and imaginary parts satisfy

$$\left(\frac{\mathrm{d}\Phi}{\mathrm{d}z}\right)^2 = m^2 + \frac{A''}{A}, \qquad \frac{2A'}{A} = -\frac{\Phi''}{\Phi'}$$

WKB approximation states

$$m^2 \gg \frac{A''}{A}, \qquad \Phi = \pm \int m(z)\,\mathrm{d}z\,, \qquad A(z) = A_o \left[\frac{m_o}{m(z)}\right]^{1/2}$$

From scaling analysis, WKB approximation is justified in large-wavenumber limit

$$m^2 H^2 \gg 1$$

This only holds for **hydrostatic IGW (near-inertial limit)** with

$$\frac{H}{L} \ll 1, \qquad \frac{|\boldsymbol{k}_h|}{m} \ll 1$$

In this limit, the approximated dispersion relation becomes

$$\omega^2 \approx f^2 + N^2 \frac{|\boldsymbol{k}_h|^2}{m^2} \ll N^2$$

The hydrostatic limit simplifies the buoyancy equation with the Boussinesq approximation

$$0 = -\frac{1}{\rho_o}\frac{\partial p'}{\partial z} + b, \qquad \left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial^2 w}{\partial z^2} + N^2 \nabla_h^2 w = 0$$

For a steady stratification, a propagating wave will conserve its frequency with $m \propto N$. For a steady state wave field, if the medium only varies in z-direction, the energy equation gives

$$\nabla \cdot \boldsymbol{F}_{\mathrm{E}} = \nabla \cdot \bar{E}\boldsymbol{c}_g = 0, \qquad \frac{\partial}{\partial z}\left(\bar{E} c_g^z\right) = 0$$

The approximated dispersion relation gives

$$c_g^z = \frac{\partial \omega}{\partial m} = -\frac{\omega^2 - f^2}{\omega m}, \qquad \bar{E} \propto m \propto N$$

For near-inertial waves, the energy is dominated by the kinetic energy. From continuity, we have the scaling relation

$$|\boldsymbol{u}_h|^2 \sim \frac{m^2 A^2}{k_h^2} \gg w^2, \qquad \bar{E} \sim |\boldsymbol{u}_h|^2 \sim m^2 A^2 \propto m$$

This implies the shoaling of IGWs as propagating into regions of enhanced stratification.

Ray tracing equation

For a medium that slowly varies with time or space, consider a solution of the form

$$A(\boldsymbol{x}, t) e^{i\Phi(\boldsymbol{x},t)}, \qquad \omega = -\frac{\partial \Phi}{\partial t}, \qquad \boldsymbol{k} = \nabla\Phi$$

The local wavenumber and frequency have the relation

$$\frac{\partial \boldsymbol{k}}{\partial t} = \frac{\partial}{\partial t}(\nabla\Phi) = \nabla\left(\frac{\partial \Phi}{\partial t}\right) = -\nabla\omega$$

Consider a general dispersion relation given by medium properties $\lambda_1, \cdots \lambda_n$

$$W(\boldsymbol{k}; \lambda_1, \cdots, \lambda_n) = 0$$

For IGWs, the expression is

$$W(\boldsymbol{k}; N^2, f^2) = \sqrt{N^2 \frac{|\boldsymbol{k}_h|^2}{|\boldsymbol{k}|^2} + f^2 \frac{m^2}{|\boldsymbol{k}|^2}}$$

The evolution of wavenumber $\boldsymbol{k}$ thus follows

$$\frac{\partial \boldsymbol{k}}{\partial t} = -\nabla \omega = -\left[\frac{\partial W}{\partial \boldsymbol{k}} \cdot \nabla \boldsymbol{k} + \sum_n \frac{\partial W}{\partial \lambda_n} \nabla \lambda_n\right] = -\boldsymbol{c}_g \cdot \nabla \boldsymbol{k} - \sum_n \frac{\partial W}{\partial \lambda_n} (\nabla \lambda_n)$$

For a vertically varying medium, horizontal wavenumbers are conserved. The evolution of ω is similarly derived as

$$\frac{\partial \omega}{\partial t} = \frac{\partial W}{\partial \boldsymbol{k}} \cdot \frac{\partial \boldsymbol{k}}{\partial t} + \sum_n \frac{\partial W}{\partial \lambda_n} \frac{\partial \lambda_n}{\partial t} = -\boldsymbol{c}_g \cdot \nabla \omega + \sum_n \frac{\partial W}{\partial \lambda_n} \frac{\partial \lambda_n}{\partial t}$$

For a medium without temporal variation, the angular frequency is conserved. We define the material derivative of the wave packet as

$$\frac{\mathrm{D}_g}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{c}_g \cdot \nabla$$

The ray equations can be written as

$$\frac{\mathrm{D}_g \boldsymbol{k}}{\mathrm{D}t} = -\sum_n \frac{\partial W}{\partial \lambda_n} (\nabla \lambda_n), \qquad \frac{\mathrm{D}_g \omega}{\mathrm{D}t} = \sum_n \frac{\partial W}{\partial \lambda_n} \frac{\partial \lambda_n}{\partial t}, \qquad \frac{\mathrm{D}_g \boldsymbol{x}_r}{\mathrm{D}t} = \boldsymbol{c}_g$$

Balanced Flows

- Geostrophic balance

In a rotating fluid, IGWs with $\omega < f$ cannot exist. For sub-inertial motion, consider the scales

$$x, y \sim L, \qquad z \sim H, \qquad t \sim T, \qquad u, v \sim U, \qquad w \sim \frac{UH}{L}, \qquad p \sim \rho_o U f L$$

The x-momentum equations (rotating frame) become

$$\frac{1}{fT}\frac{\partial \tilde{u}}{\partial \tilde{t}} - \tilde{v} + \frac{U}{fL}\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{u} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}, \qquad \mathrm{Ro_T}\frac{\partial \tilde{u}}{\partial \tilde{t}} - \tilde{v} + \mathrm{Ro}\,\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{u} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}$$

When the **Rossby numbers are small**, we obtain the geostrophic balance

$$-f v_g = -\frac{1}{\rho_o}\frac{\partial p}{\partial x}, \qquad f u_g = -\frac{1}{\rho_o}\frac{\partial p}{\partial y}, \qquad \boldsymbol{f} \times \boldsymbol{u}_g = -\frac{1}{\rho_o}\nabla_h p$$

The geostrophic flow field can be solved as

$$\boldsymbol{u}_g = \frac{1}{f\rho_o}\hat{\mathbf{z}} \times \nabla_h p$$

Properties of geostrophic flows

- Perpendicular to the PGF.
- Horizontal divergence is zero. The flow can be described by geostrophic stream function

$$\nabla_h \cdot \boldsymbol{u}_g = 0, \qquad v_g = \frac{\partial \psi}{\partial x}, \qquad u_g = -\frac{\partial \psi}{\partial y}$$

- The geostrophic stream function is proportional to pressure.

$$-f\frac{\partial \psi}{\partial x} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x}, \qquad \psi = \frac{p}{\rho_o f}$$

- Isobars are parallel to the streamlines. Counter-clockwise for low pressure center in NH.
- Speed of the geostrophic flow is the magnitude of the gradient of stream function

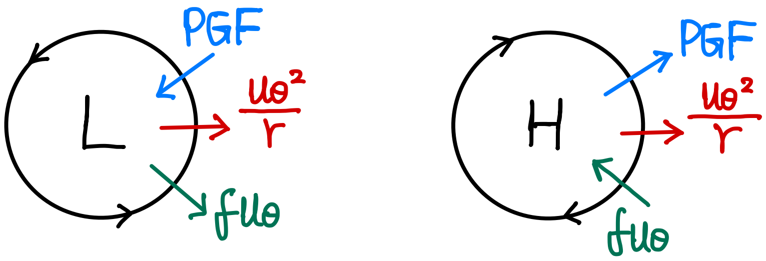
$$\sqrt{u_g^2 + v_g^2} = |\nabla \psi|$$

- Cyclo-geostrophic (gradient wind) balance in a rotating fluid

In an axisymmetric steady flow with small temporal Rossby number, the gradient wind balance is the radial momentum balance

$$-\frac{1}{\rho_o}\frac{\partial p}{\partial r} + f u_\theta + \frac{u_\theta^2}{r} = 0$$

The cyclo-geostrophic flow is weaker than the geostrophic flow in cyclones (i.e., vorticity is positive in the direction of Earth's angular velocity), and stronger in anticyclones. Low pressure center leads to a cyclonic flow.



Using the geostrophic flow, we can write

$$\frac{u_\theta^2}{r} + fu_\theta - fu_\theta^g = 0, \qquad fu_\theta^g = \frac{1}{\rho_o}\frac{\partial p}{\partial r}$$

Define the Rossby numbers as

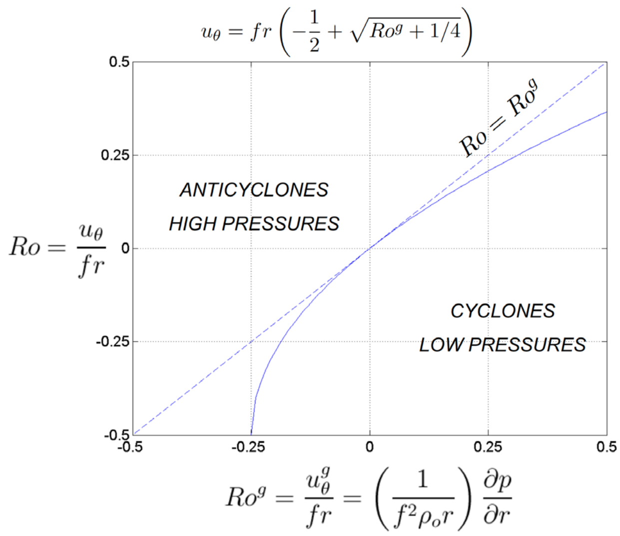
$$\mathrm{Ro} = \frac{u_\theta}{fr}, \qquad \mathrm{Ro_g} = \frac{u_\theta^g}{fr} = \frac{1}{f^2\rho_o r}\frac{\partial p}{\partial r}$$

The solution to the Rossby number is

$$\mathrm{Ro}^2 + \mathrm{Ro} - \mathrm{Ro_g} = 0$$

$$\mathrm{Ro} = \frac{-1 + \sqrt{1 + 4\mathrm{Ro_g}}}{2}$$

We select the positive root because as the radial pressure gradient goes to zero, both Rossby numbers should also go to zero.



At the critical value $\mathrm{Ro_g} = -1/4$, we have

$$\mathrm{Ro_g} = \frac{u_\theta^g}{fr} = \frac{1}{f^2\rho_o r}\frac{\partial p}{\partial r} = -\frac{1}{4}, \qquad \mathrm{Ro} = \frac{u_\theta}{fr} = -\frac{1}{2}$$

With the shallow water approximation

$$\frac{1}{\rho_o}\frac{\partial p}{\partial r} = g\frac{\partial \eta'}{\partial r} = -\frac{1}{4}f^2 r = -\Omega^2 r, \qquad \eta' = -\frac{\Omega^2}{2g}r^2 = -\bar{\eta}$$

The total water surface $\eta' + \bar{\eta}$ then is flat. The vertical vorticity in this case is

$$u_\theta = -\frac{1}{2}fr, \qquad (\nabla \times \boldsymbol{u}) \cdot \hat{\mathbf{z}} = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) = -f$$

The criterion for centrifugal (inertial) instability is $\zeta_z < -f$. In other words, balanced motions with anticyclonic vorticity $|\zeta_z| > |f|$ are not possible.

Centrifugal instability (CI)

Consider a jet in x-direction under geostrophic balance. The flow is inviscid and is invariant in the x-direction. With small perturbation u', we have

$$u = u_g(y) + u', \qquad f u_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

The momentum equation in x-direction is (conservation of absolute momentum)

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = \frac{\mathrm{D}}{\mathrm{D}t}\left(u_g + u' - fy\right) = 0$$

At $t = 0$, we have $y = y_o$ and $u' = 0$. Then it becomes

$$u_g + u' - fy = u_g(y_o) - f y_o, \qquad u' = f(y - y_o) - \left[u_g(y) - u_g(y_o)\right]$$

Small perturbation indicates that

$$u' = \left[f - u_g'(y_0)\right]\Delta y = \left(f + \zeta_g\right)\Delta y, \qquad \zeta_g = -\frac{\partial u_g}{\partial y}$$

The momentum equation in y-direction becomes

$$\frac{\mathrm{D}v}{\mathrm{D}t} + f\left(u_g + u'\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{\mathrm{D}^2 \Delta y}{\mathrm{D}t} + f\left(f + \zeta_g\right)\Delta y = 0$$

Instability occurs when the vorticity of the geostrophic flow is anticyclonic and exceeds $|f|$. For NH with $f > 0$, the condition is $\zeta_g < -f$. The quantity $f + \zeta_g$ is the absolute vorticity of the geostrophic flow.

Vorticity Dynamics

- Vorticity

Consider the perturbation in the velocity field $\boldsymbol{u}(\boldsymbol{x})$ as

$$\delta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}+\delta\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}) \approx \nabla\boldsymbol{u}\cdot\delta\boldsymbol{x}$$

The gradient (Jacobian) can be decomposed into

$$(\nabla\boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \dot{\varepsilon}_{ij} + R_{ij}$$

The vorticity vector $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

We can show that

$$\frac{1}{2}(\boldsymbol{\omega}\times\delta\boldsymbol{x})_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j = (\boldsymbol{R}\cdot\delta\boldsymbol{x})_i$$

The rotation tensor corresponds to a solid body rotation with angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}/2$. The vorticity is equal to twice the local angular velocity.

$$\boldsymbol{R}\cdot\delta\boldsymbol{x} = \frac{1}{2}\boldsymbol{\omega}\times\delta\boldsymbol{x}, \qquad R_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k$$

Absolute vorticity

In the Earth's rotating local coordinate, consider the flow has a vorticity $\boldsymbol{\omega}$. In the inertial frame, the total angular velocity, and the absolute vorticity are

$$\boldsymbol{\Omega}_a = \boldsymbol{\Omega} + \frac{\boldsymbol{\omega}}{2} = \frac{f\hat{\mathbf{k}} + \boldsymbol{\omega}}{2}, \qquad \boldsymbol{\omega}_a = f\hat{\mathbf{k}} + \boldsymbol{\omega} = 2\boldsymbol{\Omega}_a$$

- Vorticity equation

Start with the momentum equation in the rotating frame without Boussinesq approximation

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + f\hat{\mathbf{k}}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \boldsymbol{F}$$

Based on the following identities

$$\nabla\cdot\boldsymbol{u} = 0, \qquad \nabla\cdot\boldsymbol{\omega}_a = 0$$

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) + \boldsymbol{\omega}\times\boldsymbol{u}$$

$$\begin{aligned}\nabla\times(\boldsymbol{\omega}_a\times\boldsymbol{u}) &= \boldsymbol{\omega}_a(\nabla\cdot\boldsymbol{u}) - \boldsymbol{u}(\nabla\cdot\boldsymbol{\omega}_a) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}_a - (\boldsymbol{\omega}_a\cdot\nabla)\boldsymbol{u} \\ &= (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}_a - (\boldsymbol{\omega}_a\cdot\nabla)\boldsymbol{u}\end{aligned}$$

$$-\nabla\times\left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho^2}\nabla\rho\times\nabla p - \frac{1}{\rho}\nabla\times\nabla p = \frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Taking the curl of the momentum equation, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega}_a \times \boldsymbol{u}) = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}_a - (\boldsymbol{\omega}_a \cdot \nabla)\boldsymbol{u}$$

$$= -\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \boldsymbol{F} = \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \boldsymbol{F}$$

The vorticity equation is

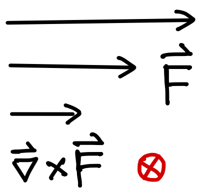
$$\frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = (\boldsymbol{\omega}_a \cdot \nabla)\boldsymbol{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nabla \times \boldsymbol{F}$$

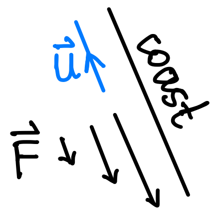
The three terms that contribute to the evolution of vorticity are

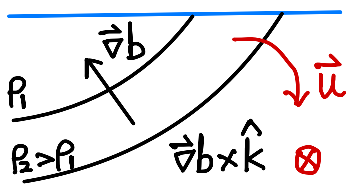
Vortex stretching / tilting	Baroclinic torque	Frictional torque
$(\boldsymbol{\omega}_a \cdot \nabla)\boldsymbol{u}$	$\frac{1}{\rho^2}\nabla\rho \times \nabla p$	$\nabla \times \boldsymbol{F}$

Frictional torque

- Consider the wind friction $\boldsymbol{F}(z)$ decaying with depth, the frictional torque points in the horizontal direction, which leads to an overturning flow.
- Consider near the coast the friction $\boldsymbol{F}(z)$ decaying towards the ocean. The frictional torque points in the downward direction, which leads to a horizontal vortex.







Baroclinic torque

Decompose the density and pressure fields as

$$\rho(\boldsymbol{x}, t) = \rho_o + \rho'(\boldsymbol{x}, t), \qquad p = p_o(z) + p'(\boldsymbol{x}, t)$$

Under Boussinesq assumption, the first-order term is

$$\frac{1}{\rho^2}\nabla\rho \times \nabla p \approx \frac{1}{\rho_o^2}\nabla\rho' \times \nabla p_o = -\frac{g}{\rho_o}\nabla\rho' \times \hat{\mathbf{k}} = \nabla_h b \times \hat{\mathbf{k}}$$

The baroclinic torque is horizontal and is non-zero only if there are lateral density gradients. As an example, consider density surfaces tilted upward. The buoyancy gradient $\nabla_h b$ points to the lower density region, and the baroclinic torque points to the other horizontal direction. The induced flow is consistent with lighter fluids moving on top of the denser fluids.

Vortex stretching / tilting visualized by a line element

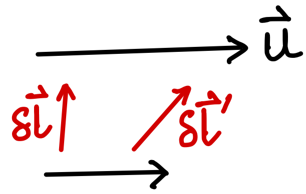
Consider a line element $\delta\boldsymbol{l}$ within the flow. At a later time $t + \delta t$, the element becomes

$$\delta\boldsymbol{l}' = \delta\boldsymbol{l} + \boldsymbol{u}(\boldsymbol{x} + \delta\boldsymbol{l})\delta t - \boldsymbol{u}(\boldsymbol{x})\delta t = \delta\boldsymbol{l} + (\delta\boldsymbol{l} \cdot \nabla)\boldsymbol{u}\delta t$$

The evolution of the line element is governed by

$$\frac{\mathrm{D}\delta \boldsymbol{l}}{\mathrm{D}t} = (\delta \boldsymbol{l} \cdot \nabla)\boldsymbol{u}$$

This shares the same form as the vortex stretching / tilting.

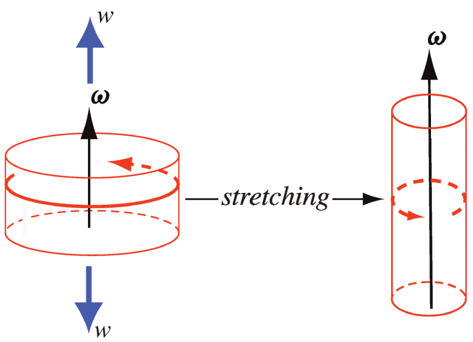


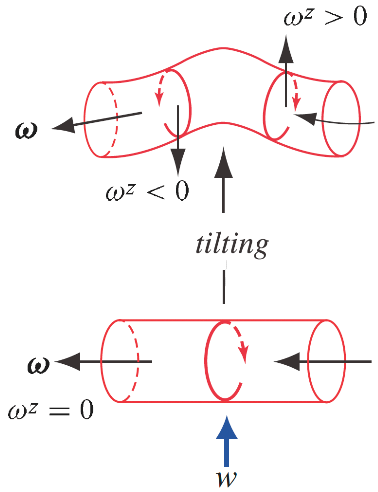
Consider an initial $\boldsymbol{\omega}_a = \omega_z \hat{\mathbf{k}}$ in a flow $w = w(z)$, the vortex stretching is in z-direction

$$\frac{\mathrm{D}\omega_z}{\mathrm{D}t} = \omega_z \frac{\mathrm{d}w}{\mathrm{d}z}$$

Now consider a shear flow $u = u(z)$, the vorticity equation leads to vortex tilting

$$\frac{\mathrm{D}\omega_x}{\mathrm{D}t} = \omega_z \frac{\mathrm{d}u}{\mathrm{d}z}, \qquad \frac{\mathrm{D}\omega_z}{\mathrm{D}t} = 0$$





➢ Vorticity equation in low Rossby number limit

Taylor-Proudman effect

The Rossby number compares the vertical vorticity ζ with Coriolis parameter f

$$\frac{\zeta}{f} = \frac{1}{f}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \sim \frac{U}{fL}$$

With Ro $\ll 1$, we have

$$\boldsymbol{\omega}_a \cdot \hat{\mathbf{k}} = f + \zeta \approx f, \qquad \boldsymbol{\omega}_a \approx f\hat{\mathbf{k}}, \qquad \frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = \mathbf{0}$$

This implies that for an inviscid fluid with no baroclinic torque, the flow does not vary in the vertical direction in the limit Ro $\ll 1$

$$\frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = \boldsymbol{\omega}_a \cdot \nabla\boldsymbol{u} \approx f\frac{\partial \boldsymbol{u}}{\partial z} \approx \mathbf{0}, \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0$$

This describes the Taylor-Proudman effect, or the gyroscopic rigidity. The effects of rotation have provided a stiffening of the fluid in the vertical.

Thermal wind balance

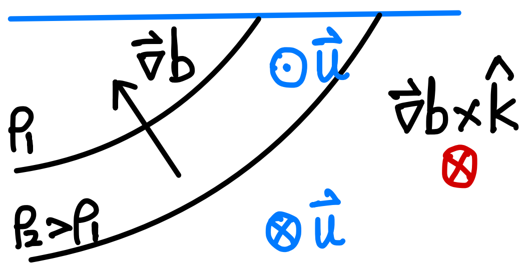
Still in the limit of Ro $\ll 1$, if there is a baroclinic torque, we have

$$f\frac{\partial \boldsymbol{u}}{\partial z} + \nabla_h b \times \hat{\mathbf{k}} = \mathbf{0}$$

This describes thermal wind balance

$$\frac{\partial b}{\partial y} + f\frac{\partial u}{\partial z} = 0, \qquad \frac{\partial b}{\partial z} + f\frac{\partial v}{\partial z} = 0$$

The baroclinic torque balances the vortex tilting due to planetary vorticity. For the example of density surfaces tilted upward, in order to counteract the effect due to baroclinic torque, the balanced flow should tilt the vorticity vector to the opposite direction.



- Potential vorticity (PV)

The absolute vorticity $\boldsymbol{\omega}_a$ is not conserved even with no frictional torque and in a homogeneous medium, since there exists the vortex stretching term. However, the PV is conserved.

PV in shallow water system

Denote the height of the water column as h. For a vertically uniform velocity field $\boldsymbol{u}_h(x,y)$, the absolute vorticity is

$$\boldsymbol{\omega}_a = (f + \zeta)\,\hat{\mathbf{k}}, \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The momentum equations in the horizontal directions are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\partial \eta}{\partial x} + F_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\partial \eta}{\partial y} + F_y$$

We can combine them and obtain the equation for vorticity ζ

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u}\cdot\nabla\zeta + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

The governing equation for absolute vorticity is

$$\frac{\mathrm{D}(f+\zeta)}{\mathrm{D}t} = -(f+\zeta)(\nabla_h\cdot\boldsymbol{u}) + (\nabla\times\boldsymbol{F})\cdot\hat{\mathbf{k}}$$

With the continuity equation

$$\frac{\mathrm{D}h}{\mathrm{D}t} = -h(\nabla_h \cdot \boldsymbol{u}), \qquad \nabla_h \cdot \boldsymbol{u} = -\frac{1}{h}\frac{\mathrm{D}h}{\mathrm{D}t}$$

we can further obtain

$$\frac{1}{h}\frac{\mathrm{D}(f+\zeta)}{\mathrm{D}t} - \frac{f+\zeta}{h^2}\frac{\mathrm{D}h}{\mathrm{D}t} = \frac{(\nabla \times \boldsymbol{F}) \cdot \hat{\mathbf{k}}}{h}, \qquad \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{f+\zeta}{h}\right) = \frac{(\nabla \times \boldsymbol{F}) \cdot \hat{\mathbf{k}}}{h}$$

We can define the potential vorticity for a shallow water system as

$$q = \frac{f+\zeta}{h}, \qquad \frac{\mathrm{D}q}{\mathrm{D}t} = \frac{(\nabla \times \boldsymbol{F}) \cdot \hat{\mathbf{k}}}{h}$$

Consider a spinning water column, and its angular velocity is $\alpha = (f+\zeta)/2$. The conservation of mass states that

$$m = \rho_o \pi R^2 h = \text{const.}$$

Without frictional torque, the angular momentum is conserved, which is

$$L_z = I\alpha = \frac{mR^2}{2} \cdot \frac{f+\zeta}{2} = \frac{m^2}{2\rho_o \pi h}\frac{f+\zeta}{2} = \text{const.}$$

The conservation of mass and angular momentum lead to

$$\frac{f+\zeta}{h} = \text{const.}$$

The water column in a shallow water has higher potential to create vorticity. As it moves to the deeper ocean, the column is stretched and thus spins up.

Ertel PV in stratified fluid

Define the Ertel PV as

$$q = \boldsymbol{\omega}_a \cdot \nabla b$$

It describes the component of absolute vorticity in the direction ∇b. Since $\nabla_h b \times \hat{\mathbf{k}}$ is always perpendicular to ∇b, the baroclinic torque does not contribute to q. Note that

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}_a \cdot \nabla b) = \nabla b \cdot \frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} + \boldsymbol{\omega}_a \cdot \frac{\mathrm{D}}{\mathrm{D}t}(\nabla b)$$

From the vorticity equation, we first have

$$\nabla b \cdot \frac{\mathrm{D}\boldsymbol{\omega}_a}{\mathrm{D}t} = \nabla b \cdot (\boldsymbol{\omega}_a \cdot \nabla)\boldsymbol{u} + \nabla b \cdot (\nabla \times \boldsymbol{F})$$

From the equation of state, we can obtain the buoyancy equation

$$b = g[\alpha(T - T_o) - \beta(S - S_o)], \qquad \frac{\mathrm{D}b}{\mathrm{D}t} = \alpha_g \kappa_T \nabla^2 T - \beta g \kappa_S \nabla^2 S = \mathcal{D}$$

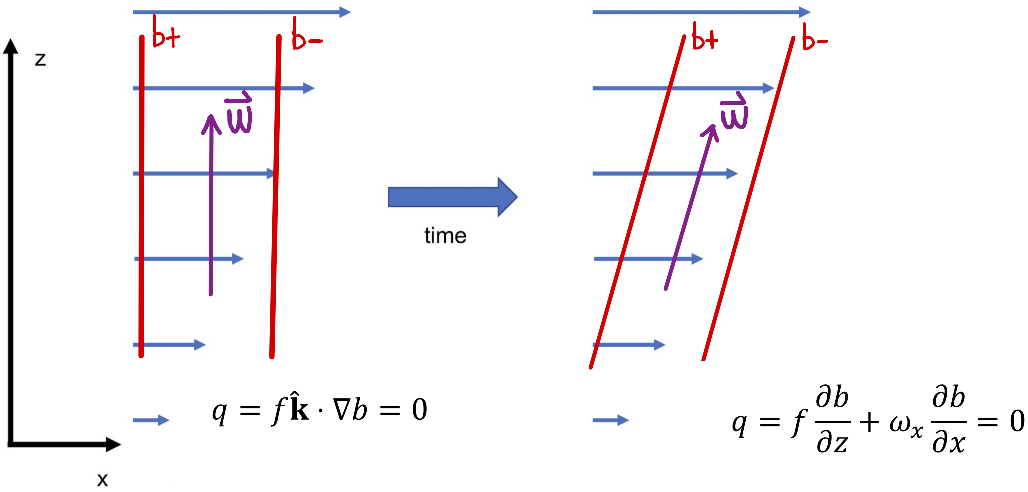
where all the diabatic processes are included in $\mathcal{D}$. The gradient of buoyancy equation gives

$$\frac{\mathrm{D}}{\mathrm{D}t}(\nabla b) = \nabla \mathcal{D} - \nabla \boldsymbol{u} \cdot \nabla b$$

The term $-\nabla\boldsymbol{u}\cdot\nabla b$ denotes the generation of ∇b by stretching / tilting of isopycnals. Finally, the governing equation for Ertel PV is

$$\frac{\mathrm{D}q}{\mathrm{D}t}=\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}_a\cdot\nabla b)=\nabla b\cdot(\nabla\times\boldsymbol{F})+\boldsymbol{\omega}_a\cdot\nabla\mathcal{D}$$

The advective processes such as tilting of isopycnals and vortex stretching do not contribute to the change of q. Only non-conservative frictional and diabatic processes contribute.



Summary of potential vorticity

- For inviscid and adiabatic flows, PV is conserved following fluid parcels, but it is not necessarily true for vorticity.
- PV is advected by the flow as a dynamical tracer. It is not a passive tracer because its distribution affects the flow.
- Invertibility principle: If the flow satisfies a balance relation, then all dynamic variables $(\boldsymbol{u},h,p,b)$ can be derived from the PV field. PV plays an analogous role to the vertical vorticity in an inviscid 2D horizontal flow.

Geostrophic Theory

- Geostrophic adjustment

The geostrophic adjustment problem describes how an initially unbalanced flow adjusts to a geostrophically balanced flow. Consider a shallow water system with no frictional torque

$$q = \frac{f+\zeta}{h}, \qquad \frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

The developed flow is under geostrophic balance

$$-fu = g\frac{\partial \eta}{\partial y}, \qquad fv = g\frac{\partial \eta}{\partial x}, \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{g}{f}\nabla_h^2 \eta$$

Define the barotropic Rossby radius of deformation as (with mean ocean depth H)

$$L_r = \frac{\sqrt{gH}}{f}$$

It is the distance traveled over timescale $1/f$ with the shallow water surface wave speed. From scaling analysis, consider the amplitude $\eta \ll H$, we have

$$U \sim \frac{g\eta}{fL}, \qquad \frac{|\boldsymbol{u}\cdot\nabla q|}{|\partial q/\partial t|} \sim \frac{U}{fL} \sim \frac{g\eta}{f^2L^2} \sim \frac{\eta}{H} \ll 1, \qquad \frac{\zeta}{f} \sim \frac{U}{fL} \sim \frac{\eta}{H} \ll 1$$

This implies that PV is approximately conserved at a fixed location. For a linear flow, we have

$$q = \frac{f+\zeta}{H+\eta} \approx \frac{f}{H}\left(1+\frac{\zeta}{f}\right)\left(1-\frac{\eta}{H}\right) = \frac{f}{H} + q', \qquad q' = \frac{\zeta}{H} - \frac{f\eta}{H^2}$$

For constant f, we can only focus on the PV anomaly q' from the background PV $\bar{q}$. Given an initial q_0', the governing equation for the free surface η becomes

$$\frac{g}{f}\nabla_h^2\eta - \frac{f\eta}{H} = q_0'H, \qquad \nabla_h^2\eta - \frac{\eta}{L_r^2} = \frac{fH}{g}q_0'$$

From the PV field, we can solve for η and then obtain $\boldsymbol{u}$ from the geostrophic balance.

Green's function

Consider the PV field is invariant in y-direction. The Green's function $G(x;x')$ satisfies

$$\frac{\mathrm{d}^2G}{\mathrm{d}x^2} - \frac{G}{L_r^2} = \delta(x-x')$$

The continuity of G and discontinuity of $\mathrm{d}G/\mathrm{d}x$ at $x = x'$ give the solution

$$G(x;x') = -\frac{L_r}{2}\exp\left(-\frac{|x-x'|}{L_r}\right)$$

Geostrophic adjustment for an initial step

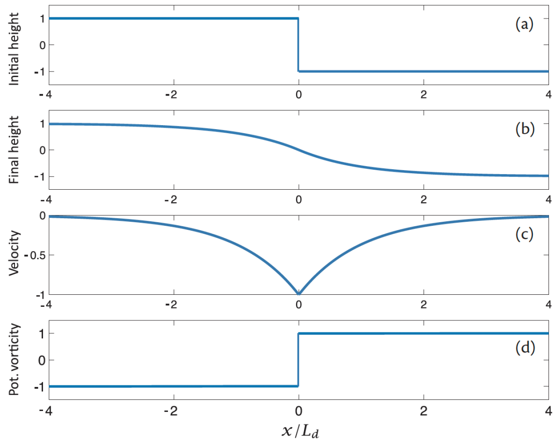
Consider an initial step surface $\eta_o(x) = -\eta_o\mathrm{sgn}(x)$ with $\boldsymbol{u}_h = \mathbf{0}$. We need to solve

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} - \frac{\eta}{L_r^2} = \frac{\eta_o}{L_r^2}\mathrm{sgn}(x)$$

The solution is obtained as

$$\eta(x) = -\eta_o \mathrm{sgn}(x) \cdot \left(1 - e^{-\frac{|x|}{L_r}}\right)$$

$$v(x) = \frac{g}{f}\frac{\partial \eta}{\partial x} = -\frac{g\eta_o}{fL_r} e^{-|x|/L_r}$$



The potential vorticity conservation constrains the influence of the adjustment to within a deformation radius L_r of the initial disturbance. After timescale T, the waves will experience the effect of rotation, generate a Coriolis force that eventually balances the PGF.

Energetics of geostrophic adjustment

The general expression for the total potential energy (per unit length in y) is

$$\mathrm{PE} = \rho_o \int_{-\infty}^{+\infty} g(H+\eta) \cdot \frac{H+\eta}{2}\,\mathrm{d}x = \frac{\rho_o g}{2} \int_{-\infty}^{+\infty} (H^2 + 2H\eta + \eta^2)\,\mathrm{d}x$$

Since the average of η is zero, and the first term denotes the background, the available potential energy (APE) is

$$\mathrm{APE} = \frac{\rho_o g}{2} \int_{-\infty}^{+\infty} \eta^2\,\mathrm{d}x$$

The initial APE and the perturbation after the geostrophic adjustment are calculated as

$$\mathrm{APE}_0 = \frac{\rho_o g}{2} \int_{-\infty}^{\infty} \eta_o^2\,\mathrm{d}x, \qquad \Delta\mathrm{APE} = \frac{\rho_o g}{2} \int_{-\infty}^{\infty} [\eta^2(x) - \eta_o^2]\,\mathrm{d}x = -\frac{3}{2}\rho_o g \eta_o^2 L_r$$

The final kinetic energy is

$$\Delta\mathrm{KE} = \frac{\rho_o H}{2} \int_{-\infty}^{\infty} v^2(x)\,\mathrm{d}x = \frac{1}{2}\frac{\rho_o g^2 H \eta_o^2}{f^2 L_r} = \frac{1}{2}\rho_o g \eta_o^2 L_r$$

Therefore, only $1/3$ of ΔAPE is converted to ΔKE. The remaining part is radiated away by Poincaré waves and lost to infinity. On the contrary, in the non-rotating case all APE is released.

- Poincaré wave

The transient geostrophic adjustment of the flow triggered Poincaré waves. In small amplitude limit $\eta/H \ll 1$, the linearized shallow water equations are

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial \eta}{\partial x}, \qquad \frac{\partial v}{\partial t} = -fu - g\frac{\partial \eta}{\partial y}, \qquad \frac{\partial \eta}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

To obtain an equation for surface height η, take the divergence of the horizontal momentum equations, and take the time derivative of the continuity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=f\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-g\nabla_h^2\eta,\qquad \frac{\partial^2\eta}{\partial t^2}=-H\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

We thus obtain

$$\frac{\partial^2\eta}{\partial t^2}=-fH\zeta+gH\nabla_h^2\eta$$

The conservation of PV anomaly is

$$q'(\boldsymbol{x})=\frac{\zeta}{H}-\frac{f\eta}{H^2}=\text{const.},\qquad \zeta=q'H+\frac{f\eta}{H}$$

We thus obtain

$$\frac{\partial^2\eta}{\partial t^2}-gH\nabla_h^2\eta+f^2\eta=-fH^2q'(\boldsymbol{x})$$

The particular solution $\eta_p(\boldsymbol{x})$ represents the free surface of the steady geostrophic flow given the PV anomaly $q'(\boldsymbol{x})$, which satisfies

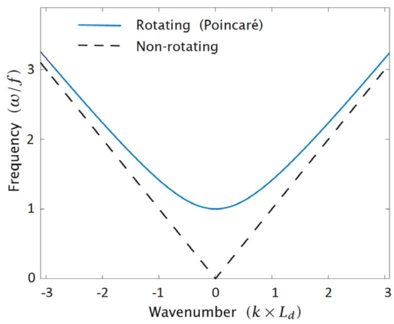
$$\nabla_h^2\eta_p-\frac{\eta_p}{L_r^2}=\frac{fH}{g}q'$$

The homogeneous part $\eta(\boldsymbol{x},t)$ satisfies the Poincaré wave equation

$$\frac{\partial^2\eta}{\partial t^2}-gH\nabla_h^2\eta+f^2\eta=0$$

Only balanced, geostrophic flow has the PV anomaly, while Poincaré waves have $q'=0$. The Poincaré waves adjust the initial surface not under geostrophic balance, i.e., $\eta_0-\eta_p$.

Dispersion relation of Poincaré wave



In the Fourier domain, we obtain the dispersion relation

$$\omega^2=f^2+gH(k^2+l^2),\qquad \left(\frac{\omega}{f}\right)^2=1+|\boldsymbol{k}_h|^2L_r^2$$

This is similar to the near-inertial (hydrostatic) IGW, since we also assume small aspect ratio $H/L\ll 1$. In short wavelength limit $\lambda\ll L_r$, the wave is non-dispersive, with phase velocity the same as the shallow water wave without rotation

$$|\boldsymbol{k}_h|L_r\gg 1,\qquad \lambda\ll L_r,\qquad \omega=\sqrt{gH}\,|\boldsymbol{k}_h|,\qquad c=\sqrt{gH}$$

In the long wavelength limit, we recover the inertial oscillation

$$|\boldsymbol{k}_h|L_r\ll 1,\qquad \lambda\gg L_r,\qquad \omega=f$$

Group velocity of Poincaré wave

$$c_g^x=gH\frac{k}{\omega},\qquad c_g^y=gH\frac{l}{\omega}$$

Therefore, the short waves radiate most of the energy

$$c_g = \sqrt{gH} \quad \text{with } \lambda \ll L_r, \qquad c_g = \sqrt{gH}\,|\boldsymbol{k}_h| L_r \ll \sqrt{gH} \quad \text{with } \lambda \gg L_r$$

Polarization relation of Poincaré wave

$$v_o = \frac{if\omega - gHlk}{gHl^2 - \omega^2} u_o, \qquad \eta_o = H \frac{ilf - \omega k}{gHl^2 - \omega^2} u_o$$

➢ Kelvin wave

Another type of waves without PV anomaly is the Kelvin wave. It has one component of the fluid velocity equal to zero:

- Equatorial Kelvin wave: No meridional flow, $v = 0$
- Coastal Kelvin wave: No normal flow, $\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$

Coastal Kelvin wave

Consider a NS coastline at $x = 0$ which implies $u = 0$. The governing equations become

$$0 = fv - g\frac{\partial \eta}{\partial x}, \qquad \frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}, \qquad \frac{\partial \eta}{\partial t} = -H\frac{\partial v}{\partial y}$$

Similarly, we can obtain the Kelvin wave equation and its phase speed

$$\frac{\partial^2 \eta}{\partial t^2} - gH\frac{\partial^2 \eta}{\partial y^2} = 0, \qquad c = \sqrt{gH}$$

The general wave solution has the following form

$$\eta(x, y, t) = G(x)\, \tilde{F}(y, t), \qquad \tilde{F}(y, t) = F(y - ct) + F(y + ct)$$

This shows that Kelvin waves propagate parallel to the coastline. In the normal direction, the wave structure is solved from combining the momentum equations as

$$\frac{\partial v}{\partial t} = \frac{g}{f}\frac{\partial^2 \eta}{\partial x \partial t} = -g\frac{\partial \eta}{\partial y}, \qquad \frac{\partial^2 \eta}{\partial x \partial t} + f\frac{\partial \eta}{\partial y} = 0$$

Substituting the solution form, we obtain

$$\eta_\pm(x, y, t) = G_\pm(x) F(y \mp ct), \qquad \frac{\mathrm{d}G_\pm}{\mathrm{d}x} = \pm\mathrm{sgn}(f)\frac{|f|}{c} G_\pm = \pm\mathrm{sgn}(f)\frac{G_\pm}{L_r}$$

For Kelvin waves propagating in $\pm y$-direction, we have

$$\eta_+(x) = G_+(x)\, F(y - ct), \qquad G_+(x) = \exp\left[\mathrm{sgn}(f)\frac{x}{L_r}\right]$$

$$\eta_-(x) = G_-(x)\, F(y + ct), \qquad G_-(x) = \exp\left[-\mathrm{sgn}(f)\frac{x}{L_r}\right]$$

Consider the ocean in the region $x < 0$. The condition at $x \to -\infty$ implies

- Northern Hemisphere: $\eta_+(x, y, t)$, Kelvin waves propagating with **coast to the right**.
- Southern Hemisphere: $\eta_-(x, y, t)$, Kelvin waves propagating with **coast to the left**.

The PV anomaly for the Kelvin wave is

$$q' = \frac{\zeta}{H} - \frac{f\eta}{H^2} = \frac{1}{H}\frac{\partial v}{\partial x} - \frac{f\eta}{H^2} = \frac{g}{fH}\left(\frac{\partial^2 \eta}{\partial x^2} - \frac{\eta}{L_r^2}\right) = 0$$

Kelvin waves do not carry PV anomaly, even though v is geostrophic.

➢ Barotropic (planetary) Rossby wave

The existence of Rossby waves require a background PV field with spatial gradient

$$q(x, y, t) = \bar{q}(x, y) + q'(x, y, t)$$

In the shallow water system, with no background flow we can write

$$\bar{q}(x, y) = \frac{f(y)}{H(x, y)}$$

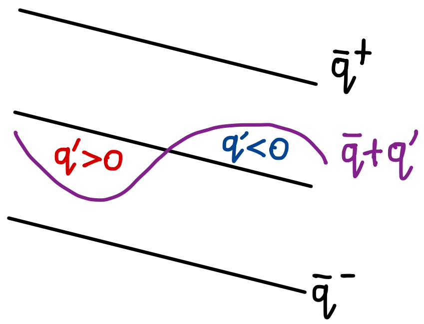
PV conservation for an inviscid and adiabatic flow gives

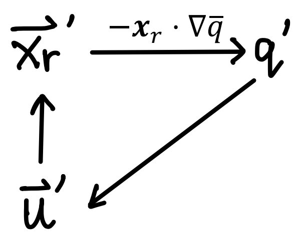
$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\mathrm{D}\bar{q}}{\mathrm{D}t} + \frac{\mathrm{D}q'}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}q'}{\mathrm{D}t} = -\frac{\mathrm{D}\bar{q}}{\mathrm{D}t} = -\boldsymbol{u} \cdot \nabla\bar{q}$$

This implies that q' is caused by the advection of background $\bar{q}$. Consider a fluid parcel with location $\boldsymbol{x}_r$, the PV anomaly equation leads to

$$\frac{\mathrm{D}q'}{\mathrm{D}t} = -\frac{\mathrm{D}\boldsymbol{x}_r}{\mathrm{D}t} \cdot \nabla\bar{q}, \qquad q' = -\boldsymbol{x}_r \cdot \nabla\bar{q}$$

Displacements of fluid parcels parallel to $\nabla\bar{q}$ will induce a PV anomaly q', which is associated with a flow anomaly $\boldsymbol{u}'$ from PV invertibility, and $\boldsymbol{u}'$ further changes the parcel displacement. This feedback can give rise to a restoring force that leads to Rossby waves.





Barotropic Rossby wave

Under the β-plane approximation with constant H and $\beta L \ll f_o$, the background PV field is

$$\bar{q} = \frac{f_o + \beta y}{H}, \qquad \nabla\bar{q} = \frac{\beta}{H}\hat{\mathbf{j}}$$

At low Rossby number, the flow is nearly geostrophic with weak amplitudes. For PV anomaly, taking the first-order terms gives

$$q' = \frac{\zeta}{H} - \frac{f_o\eta}{H^2}$$

The first term is the vorticity, while the second term is the thickness anomaly, or the stretching term. Consider the flow is geostrophic, we have

$$f_o u = -g\frac{\partial \eta}{\partial y}, \qquad f_o v = g\frac{\partial \eta}{\partial x}$$

Introduce the geostrophic stream function ψ and we obtain

$$\psi = \frac{g\eta}{f_o}, \qquad u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \qquad \zeta = \nabla^2\psi$$

The governing equation for PV anomaly then becomes

$$q' = \frac{1}{H}\left(\nabla^2\psi - \frac{\psi}{L_r^2}\right), \qquad \frac{\mathrm{D}q'}{\mathrm{D}t} = -\boldsymbol{u}\cdot\nabla\bar{q} = -\frac{v\beta}{H} = -\frac{\beta}{H}\frac{\partial \psi}{\partial x}$$

The barotropic Rossby wave equation, or the QG-PV equation, is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla^2\psi - \frac{\psi}{L_r^2}\right) + \beta\frac{\partial \psi}{\partial x} = 0, \qquad q' = \nabla^2\psi - \frac{\psi}{L_r^2}$$

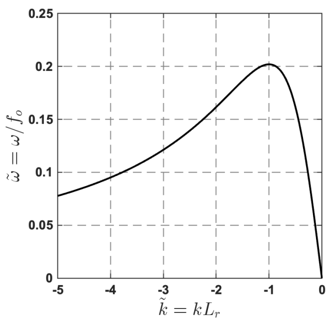
q' is called the barotropic quasi-geostrophic (QG) PV, although it has a unit of vorticity. The material derivative of q' can be further denoted as

$$\frac{\mathrm{D}q'}{\mathrm{D}t} = \frac{\partial q'}{\partial t} + u\frac{\partial q'}{\partial x} + v\frac{\partial q'}{\partial y} = \frac{\partial q'}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial q'}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial q'}{\partial y} = \frac{\partial q'}{\partial t} + J(\psi, q')$$

The Jacobian $J(A, B)$ is used for simple notation, and we can thus write

$$\frac{\partial q'}{\partial t} + J(\psi, q') + \beta\frac{\partial \psi}{\partial x} = 0, \qquad \text{with} \quad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + J(\psi, \cdot)$$

Dispersion relation for barotropic Rossby wave



Consider the plane wave solution, and with $J(A, A) = 0$ we have

$$q' = \nabla^2\psi - \frac{\psi}{L_r^2} = -\left(k^2 + l^2 + \frac{1}{L_r^2}\right)\psi, \qquad J(\psi, q') = 0$$

This leads to the dispersion relation

$$\omega = -\frac{\beta k}{k^2 + l^2 + 1/L_r^2}$$

Assume $l = 0$, and the non-dimensional frequency and wavenumber are

$$\tilde{k} = kL_r, \qquad \tilde{\omega} = \frac{\omega}{f_o} = -\frac{\beta L_r}{f_o}\cdot\frac{\tilde{k}}{\tilde{k}^2 + 1}$$

The maximum frequency of barotropic Rossby waves occurs at the following condition

$$\frac{\partial \tilde{\omega}}{\partial \tilde{k}} = -\frac{\beta L_r}{f_o}\cdot\frac{\left(\tilde{k}^2 + 1\right) - 2\tilde{k}^2}{\left(\tilde{k}^2 + 1\right)^2} = 0, \qquad \tilde{k} = \pm 1$$

Consider $\tilde{\omega} > 0$, we choose $\tilde{k} = -1$ and the maximum frequency is

$$\tilde{\omega}_{\max} = \frac{\beta L_r}{2f_o} \approx 0.2, \qquad \omega_{\max} \approx 0.2\, f_o$$

This shows that barotropic Rossby waves are **sub-inertial**. This estimation is based on

$$\beta = \frac{2\Omega}{R_e}\cos\varphi_o \approx 2\times 10^{-11}\ \ \mathrm{m^{-1}\cdot s^{-1}}, \qquad L_r = \frac{\sqrt{gH}}{f_o} \approx 2000\ \ \mathrm{km}$$

Phase velocity of Rossby wave

The x-component phase velocity is

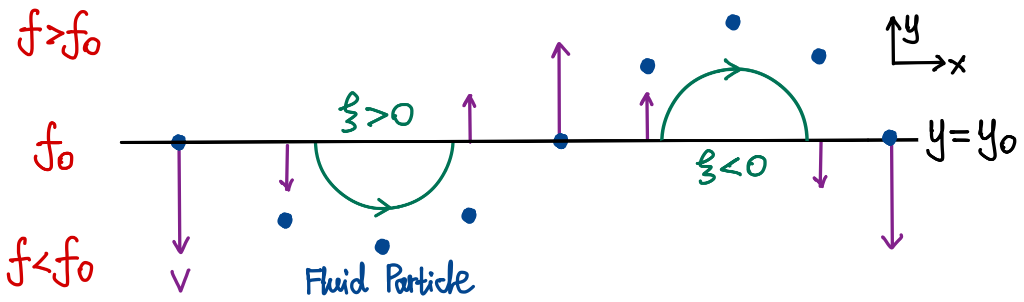
$$c_p^x = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2 + 1/L_r^2} < 0$$

This implies that the phase lines of Rossby waves, the streamlines and thus the lines of constant pressure always **propagate to the west** (-x-direction). Now consider $l = 0$ with no variation in the y-direction. The ratio of the two terms in the PV anomaly scales as

$$\frac{|\zeta/H|}{|f_o\eta/H^2|} = \frac{|\nabla^2\psi|}{|\psi/L_r^2|} \sim k^2 L_r^2$$

Under short wave limit, the vorticity term dominates PV anomaly and we have

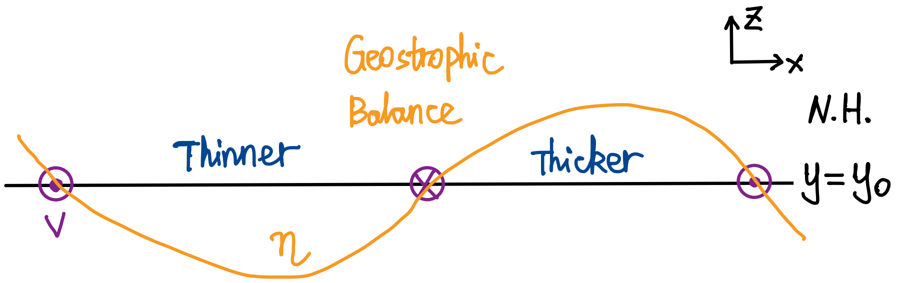
$$|k|L_r \gg 1, \qquad q' \approx \frac{\zeta}{H}, \qquad q \approx \frac{f+\zeta}{H}$$



Under long wave limit, the thickness anomaly (stretching) term dominates and we have

$$|k|L_r \ll 1, \qquad q' \approx -\frac{f_o\eta}{H^2}, \qquad q \approx \frac{f}{H} - \frac{f_o\eta}{H^2} \approx \frac{f}{H+\eta}$$

In both cases, the induced vorticity or geostrophic flows displace the fluid parcels in such a way that shifts the whole pattern westward over time.



Group velocity of barotropic Rossby wave

Assume $l = 0$ with no variation in the y-direction. The group velocity is calculated as

$$c_g^x = \frac{\partial \omega}{\partial k} = -\beta \cdot \frac{L_r^{-2} - k^2}{(k^2 + L_r^{-2})^2}$$

Under short wave limit, energy propagates **eastward** and the group velocity is very small.

$$|k|L_r \gg 1, \qquad c_g^x \approx \frac{\beta}{k^2} \ll \beta L_r^2$$

Under long wave limit, energy propagates **westward**. The long waves are non-dispersive, and the group velocity is fast with rapid energy transfer.

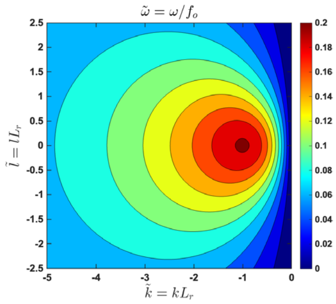
$$|k|L_r \ll 1, \qquad c_g^x \approx c_p^x \approx -\beta L_r^2$$

2D Rossby wave

The contour of ω in the k-l plane are displaced circles, which leads to **anisotropic** behavior.

$$\omega = -\frac{\beta k}{k^2 + l^2 + 1/L_r^2}$$

For a fixed wavenumber amplitude $|\boldsymbol{k}|$, its frequency ω depends on azimuth, also for phase and group velocities. In contrast, Poincaré waves are isotropic.



Wind-driven long Rossby wave modes

In terms of the spin-up of ocean circulation by winds, the initially generated Rossby waves are baroclinic with L_r replaced by the baroclinic Rossby radius of deformation $L_r^{\mathrm{bc}} \approx 50 - 100$ km at mid-latitudes. However, the winds setup perturbations with length scale $L_{\mathrm{wind}} \approx 1000$ km, which excites long Rossby wave modes that **propagate energy westward** towards the western boundary of the ocean basin.

Western intensification

Near the western boundary, energy is fluxed in by long waves and fluxed out by **reflected short waves**. Due to the difference in group velocities, energy piles up on the western boundary and thus leads to the western intensification of flows in this region.

- Regular perturbation expansion analysis

To derive the QG equation, consider the following small parameters are in the same order

$$\mathrm{Ro} = \frac{U}{fL} \sim \frac{\eta_o}{H} \sim \frac{\beta L}{f} \sim \varepsilon$$

We first obtain the non-dimensional shallow water equation using scales like L, η_o, U.

$$\mathrm{Ro}\left(\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{u}\right) - \tilde{v}\left(1 + \frac{\beta L}{f_o}\tilde{y}\right) = -\frac{\partial \tilde{\eta}}{\partial \tilde{x}}$$

$$\mathrm{Ro}\left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{v}\right) + \tilde{u}\left(1 + \frac{\beta L}{f_o}\tilde{y}\right) = -\frac{\partial \tilde{\eta}}{\partial \tilde{y}}$$

$$\frac{\eta_o}{H}\left(\frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\eta}\right) = -\left(1 + \frac{\eta_o}{H}\tilde{\eta}\right)\tilde{\nabla} \cdot \tilde{\boldsymbol{u}}$$

Based on the following expansions

$$\tilde{u} = \tilde{u}^{(0)} + \varepsilon\tilde{u}^{(1)} + \varepsilon^2\tilde{u}^{(2)} + \cdots$$

$$\tilde{v} = \tilde{v}^{(0)} + \varepsilon\tilde{v}^{(1)} + \varepsilon^2\tilde{v}^{(2)} + \cdots$$

$$\tilde{\eta} = \tilde{\eta}^{(0)} + \varepsilon\tilde{\eta}^{(1)} + \varepsilon^2\tilde{\eta}^{(2)} + \cdots$$

The leading order $O(1)$ equations give the background geostrophic flow

$$\tilde{v}^{(0)} = \frac{\partial \tilde{\eta}^{(0)}}{\partial \tilde{x}}, \qquad \tilde{u}^{(0)} = -\frac{\partial \tilde{\eta}^{(0)}}{\partial \tilde{y}}, \qquad \tilde{\nabla} \cdot \tilde{\boldsymbol{u}}^{(0)} = 0$$

The first order $O(\varepsilon)$ equations are

$$\varepsilon\left(\frac{\partial \tilde{u}^{(0)}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}^{(0)} \cdot \tilde{\nabla}\tilde{u}^{(0)}\right) - \varepsilon\tilde{v}^{(1)} - \frac{\beta L}{f_o}\tilde{y}\tilde{v}^{(0)} = -\varepsilon\frac{\partial \tilde{\eta}^{(1)}}{\partial \tilde{x}}$$

$$\varepsilon\left(\frac{\partial \tilde{v}^{(0)}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}^{(0)} \cdot \tilde{\nabla}\tilde{v}^{(0)}\right) + \varepsilon\tilde{u}^{(1)} + \frac{\beta L}{f_o}\tilde{y}\tilde{u}^{(0)} = -\varepsilon\frac{\partial \tilde{\eta}^{(1)}}{\partial \tilde{y}}$$

$$\frac{\eta_o}{H}\left(\frac{\partial \tilde{\eta}^{(0)}}{\partial \tilde{t}} + \tilde{\boldsymbol{u}}^{(0)} \cdot \tilde{\nabla}\tilde{\eta}^{(0)}\right) = -\varepsilon\tilde{\nabla} \cdot \tilde{\boldsymbol{u}}^{(1)}$$

Denote the geostrophic and ageostrophic flows as

$$u_g = U\tilde{u}^{(0)}, \qquad v_g = U\tilde{v}^{(0)}, \qquad \eta_g = \eta_o\tilde{\eta}^{(0)}$$

$$u_{ag} = \varepsilon U\tilde{u}^{(1)}, \qquad v_{ag} = \varepsilon U\tilde{v}^{(1)}, \qquad \eta_{ag} = \varepsilon\eta_o\tilde{\eta}^{(1)}$$

Then the dimensional first order $O(\varepsilon)$ equations become

$$\frac{\mathrm{D}u_g}{\mathrm{D}t} - \beta y v_g - f_o v_{ag} = -g\frac{\partial \eta_{ag}}{\partial x}$$

$$\frac{\mathrm{D}v_g}{\mathrm{D}t} + \beta y u_g + f_o u_{ag} = -g\frac{\partial \eta_{ag}}{\partial y}$$

$$\frac{\mathrm{D}\eta_g}{\mathrm{D}t} = -H\nabla \cdot \boldsymbol{u}_{ag}, \qquad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}_g \cdot \nabla$$

The vorticity equation can be constructed from the momentum equations

$$\frac{\mathrm{D}\xi_g}{\mathrm{D}t} + \beta v_g = -f_o\left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y}\right) = \frac{f_o}{H}\frac{\mathrm{D}\eta_g}{\mathrm{D}t}$$

Using the geostrophic stream function ψ, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = \frac{f_o^2}{gH}\frac{\mathrm{D}\psi}{\mathrm{D}t}, \qquad \text{with } \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + J(\psi,\cdot)$$

This is now the same as the QG-PV equation

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{\psi}{L_r^2}\right) + J\left(\psi, \nabla^2\psi - \frac{\psi}{L_r^2}\right) + \beta\frac{\partial\psi}{\partial x} = 0$$

Now consider Rossby waves in a background flow with no variation in y-direction, we have

$$u_g = 0, \qquad \frac{\partial v_g}{\partial t} + f_o u_{ag} = 0, \qquad \frac{\partial \eta_g}{\partial t} = -H\frac{\partial u_{ag}}{\partial x}$$

The acceleration of the geostrophic flow is caused by the Coriolis force from the ageostrophic flow. The free surface change is caused by the convergence of ageostrophic flow.

Group velocity of barotropic Rossby wave

Recall that the group velocity is in the same direction of the energy flux. In the x-direction, we need to analyze the correlation $\overline{pu}$. Note that

$$p = \rho_o g\eta \approx \rho_o g\eta_g, \qquad u = u_g + u_{ag} + \cdots, \qquad u_g = -\frac{g}{f_o}\frac{\partial\eta_g}{\partial y} = 0$$

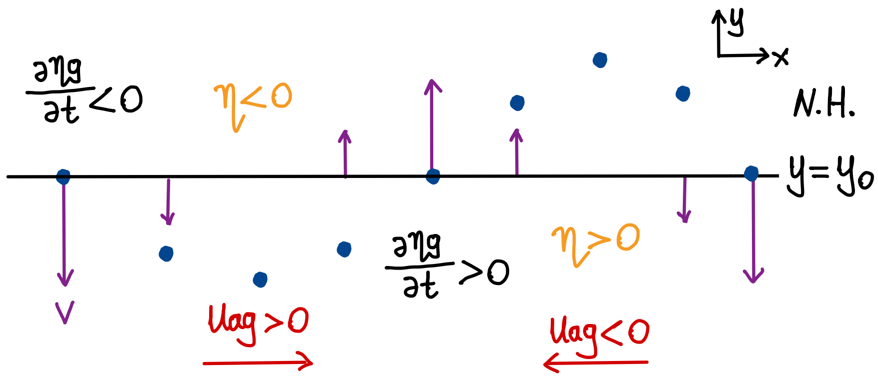
Therefore, the energy flux is determined by the correlation between displacement in the free surface and the zonal ageostrophic velocity, which is

$$\overline{pu} = \rho_o g\,\overline{\eta_g u_{ag}}$$

Under short wave limit, PV anomaly is dominated by vorticity. Therefore, we need to analyze the y-momentum equation, which gives

$$|k|L_r \gg 1, \qquad \frac{\partial v_g}{\partial t} + f_o u_{ag} = 0, \qquad \frac{\partial\zeta_g}{\partial t} = -f_o\frac{\partial u_{ag}}{\partial x}$$

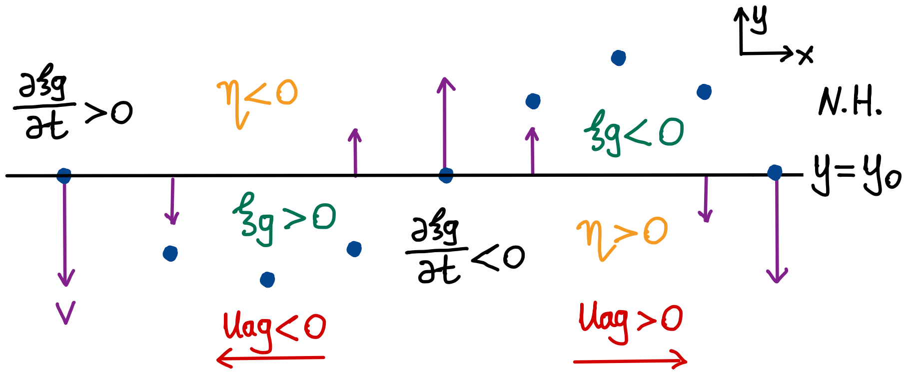
From the previous pattern of η_g, we can obtain the pattern of $\partial\zeta_g/\partial t$ and thus $\partial u_{ag}/\partial x$, which leads to the pattern of u_{ag}. This implies that the change in ζ_g is caused by the convergence of ageostrophic flow. η_g and u_{ag} share the same sign, giving $c_g^x > 0$ propagating eastward.



Under long wave limit, PV anomaly is dominated by thickness anomaly. Therefore, we need to analyze the continuity equation, which gives

$$|k|L_r \ll 1, \qquad \frac{\partial \eta_g}{\partial t} = -H\frac{\partial u_{ag}}{\partial x}$$

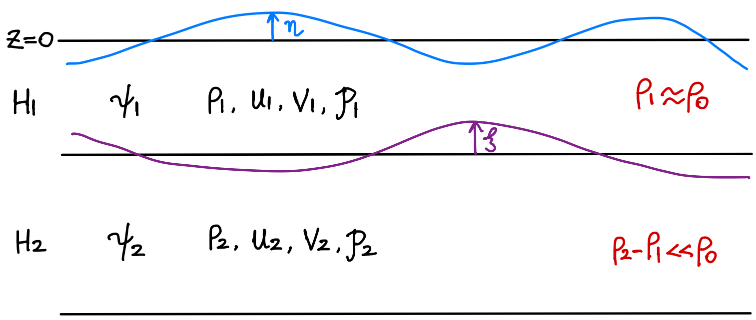
Similarly, based on the pattern of η_g, we can obtain the pattern of $\partial \eta_g/\partial t$ and thus $\partial u_{ag}/\partial x$, which leads to the pattern of u_{ag}. The change in η_g is due to the convergence of ageostrophic flow. η_g and u_{ag} share opposite signs, giving $c_g^x < 0$ propagating westward.



➢ Baroclinic Rossby wave

Two-layer model

Now we include the effects of stratification with a two-layer model. An example is the lighter and denser fluids separated by the thermocline / pycnocline in the ocean.



Consider the layers are thin compared to the horizontal length scale of the flow with $H_i \ll L$. The pressure is hydrostatic for shallow water layers. In layer 1, the PGF is calculated as

$$p_1 = p_{\text{atm}} + \rho_1 g(\eta - z), \qquad -\frac{1}{\rho_o}\nabla_h p_1 = -\frac{\rho_1}{\rho_o} g\nabla_h \eta \approx -g\nabla_h \eta$$

The Boussinesq approximation is applied with $\rho_1 \approx \rho_o$. In layer 2, we have

$$p_2 = p_{\text{atm}} + \rho_1 g(\eta + H_1 - \xi) + \rho_2 g(\xi - H_1 - z)$$

Denote the reduced gravity as g'. The PGF in layer 2 is calculated as

$$-\frac{1}{\rho_o}\nabla_h p_2 = -g\nabla_h \eta - g'\nabla_h \xi, \qquad g' = \frac{\rho_2 - \rho_1}{\rho_0} g \ll g$$

The momentum equations in both layers become

$$\frac{\mathrm{D}\boldsymbol{u}_1}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u}_1 = -g\nabla_h \eta$$

$$\frac{\mathrm{D}\boldsymbol{u}_2}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u}_2 = -g\nabla_h \eta - g'\nabla_h \xi$$

The continuity equations in each layer are

$$\frac{\mathrm{D}}{\mathrm{D}t}(H_1 + \eta - \xi) = -(H_1 + \eta - \xi)\, \nabla \cdot \boldsymbol{u}_1$$

$$\frac{\mathrm{D}}{\mathrm{D}t}(H_2 + \xi) = -(H_2 + \xi)\, \nabla \cdot \boldsymbol{u}_2$$

For a very thick lower layer with $H_2 \gg H_1$, velocity in the lower layer is very weak.

$$-\frac{1}{H_2}\frac{\mathrm{D}\xi}{\mathrm{D}t} \approx \nabla \cdot \boldsymbol{u}_2 \to 0$$

Two-layer QG-PV equation

Under the same assumptions as for barotropic Rossby waves

$$f = f_o + \beta y, \qquad \frac{\beta L}{f_o} \ll 1, \qquad \mathrm{Ro} \ll 1, \qquad \frac{\eta}{H_1} \ll 1, \qquad \frac{\xi}{H_1} \ll 1, \qquad \frac{\xi}{H_2} \ll 1$$

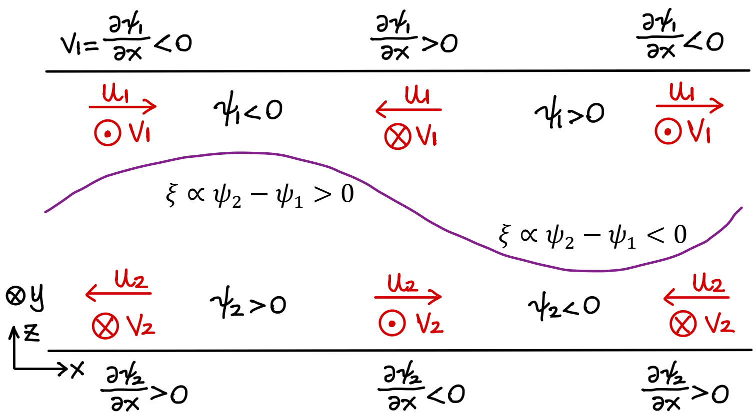
the QG approximation gives the geostrophic balance in each layer as

$$\boldsymbol{f} \times \boldsymbol{u}_1 = -g\nabla_h \eta, \qquad \boldsymbol{f} \times \boldsymbol{u}_2 = -g\nabla_h \eta - g'\nabla_h \xi$$

Using the geostrophic stream function in each layer, we have

$$\psi_1 = \frac{g}{f_o}\eta, \qquad \psi_2 = \psi_1 + \frac{g'}{f_o}\xi, \qquad \xi = \frac{f_o}{g'}(\psi_2 - \psi_1)$$

The relation between the displacement of interface ξ and stream functions are illustrated below.



The total PV in each layer is

$$q_1 = \frac{f_o + \beta y + \zeta_1}{H_1 + \eta - \xi}, \qquad q_2 = \frac{f_o + \beta y + \zeta_2}{H_2 + \xi}$$

We can decompose it into the background and perturbation PV fields as

$$\bar{q}_1 = \frac{f_o + \beta y}{H_1}, \qquad \bar{q}_2 = \frac{f_o + \beta y}{H_2}, \qquad q_1' = \frac{\zeta_1}{H_1} - \frac{f_o\eta}{H_1^2} + \frac{f_o\xi}{H_1^2}, \qquad q_2' = \frac{\zeta_2}{H_2} - \frac{f_o\xi}{H_2^2}$$

With the geostrophic stream function, the PV anomaly in each layer becomes

$$q_1' = \frac{1}{H_1}\left[\nabla^2\psi_1 - \frac{\psi_1}{L_{r,1}^2} + \frac{\psi_2 - \psi_1}{\left(L_{r,1}^{\text{bc}}\right)^2}\right], \qquad q_2' = \frac{1}{H_2}\left[\nabla^2\psi_2 - \frac{\psi_2 - \psi_1}{\left(L_{r,2}^{\text{bc}}\right)^2}\right]$$

The baroclinic Rossby radii of deformation are defined as

$$L_{r,1}^{\text{bc}} = \frac{\sqrt{g' H_1}}{f_o}, \qquad L_{r,2}^{\text{bc}} = \frac{\sqrt{g' H_2}}{f_o}$$

Now we scale the terms in PV anomaly of layer 1

$$\frac{\left|\psi_1/L_{r,1}^2\right|}{\left|\nabla^2\psi_1\right|} \sim \frac{L^2}{L_r^2} \ll 1, \qquad \frac{\left|\psi_1/L_{r,1}^2\right|}{\left|(\psi_2 - \psi_1)/\left(L_{r,1}^{\text{bc}}\right)^2\right|} \sim \frac{\left(L_{r,1}^{\text{bc}}\right)^2}{L_r^2} = \frac{g'}{g} \ll 1$$

In mid-latitudes, $H \sim 4000$ km and $L_r \sim 1000$ km. For most cases we have $L \ll L_r$, so

$$q_1' \approx \frac{1}{H_1}\left[\nabla^2\psi_1 + \frac{\psi_2 - \psi_1}{\left(L_{r,1}^{\text{bc}}\right)^2}\right], \qquad q_2' = \frac{1}{H_2}\left[\nabla^2\psi_2 - \frac{\psi_2 - \psi_1}{\left(L_{r,2}^{\text{bc}}\right)^2}\right]$$

The conservation of total PV gives

$$\frac{\mathrm{D}_1 q_1'}{\mathrm{D}t} = \frac{\partial q_1'}{\partial t} + J(\psi_1, q_1') = -\frac{\beta}{H_1}\frac{\partial\psi_1}{\partial x}$$
$$\frac{\mathrm{D}_2 q_2'}{\mathrm{D}t} = \frac{\partial q_2'}{\partial t} + J(\psi_2, q_2') = -\frac{\beta}{H_2}\frac{\partial\psi_2}{\partial x}$$

These two equations are coupled through the interface $\xi \propto \psi_2 - \psi_1$, as the change of interface leads to opposite changes of q_1' and q_2'. In the short wave limit $L \ll L_r^{\text{bc}}$, the equations become uncoupled as the coupled term in the PV anomaly is small.

Barotropic and baroclinic modes

For a simple system with $H_1 = H_2 = H$, the linearized two-layer QG-PV equations become

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_1 + \frac{\psi_2 - \psi_1}{\left(L_r^{\text{bc}}\right)^2}\right] + \beta\frac{\partial\psi_1}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_2 - \frac{\psi_2 - \psi_1}{\left(L_r^{\text{bc}}\right)^2}\right] + \beta\frac{\partial\psi_2}{\partial x} = 0$$

The equations can be decoupled by analyzing the barotropic (depth-averaged, bt) mode and the baroclinic (bc) mode separately

$$\psi_{\text{bt}} = \frac{\psi_1 + \psi_2}{2}, \qquad \psi_{\text{bc}} = \frac{\psi_1 - \psi_2}{2}$$

- When $\psi_1 = \psi_2$, we have a flat interface with $\xi = 0$. No baroclinic mode.
- When $\psi_1 = -\psi_2$, flows are opposite in each layer. No barotropic mode.

The uncoupled two-layer QG-PV equations are

$$\frac{\partial}{\partial t}(\nabla^2\psi_{\mathrm{bt}})+\beta\frac{\partial\psi_{\mathrm{bt}}}{\partial x}=0,\qquad \frac{\partial}{\partial t}\left[\nabla^2\psi_{\mathrm{bc}}-\frac{2\psi_{\mathrm{bc}}}{\left(L_r^{\mathrm{bc}}\right)^2}\right]+\beta\frac{\partial\psi_{\mathrm{bc}}}{\partial x}=0$$

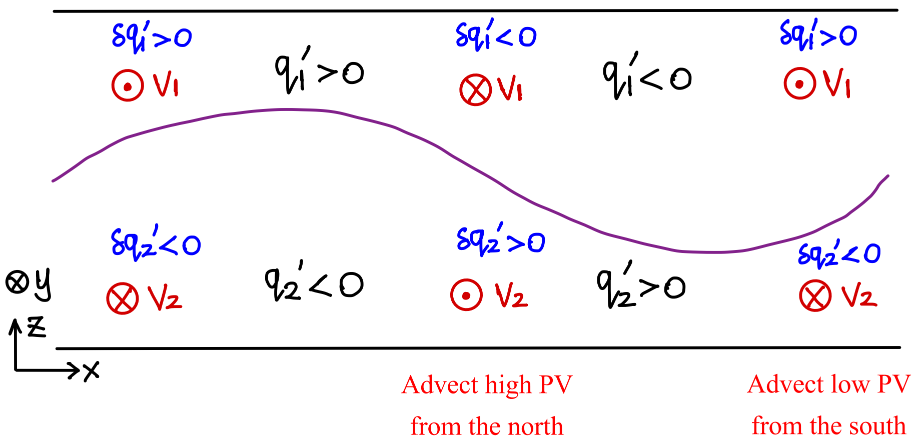
The original ψ_1 and ψ_2 are a combination of the barotropic and baroclinic modes.

Long baroclinic Rossby wave

In the long wave limit with $L \gg L_r^{\mathrm{bc}}$ ($\eta \simeq 0$), the dispersion relation is

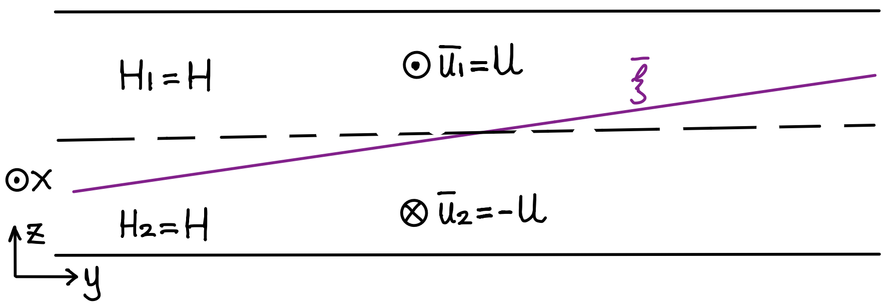
$$\frac{2i\omega}{\left(L_r^{\mathrm{bc}}\right)^2}+ik\beta=0,\qquad \omega=-\frac{\beta k}{2}\left(L_r^{\mathrm{bc}}\right)^2$$

Baroclinic modes are non-dispersive, independent of l wavenumber and propagate **westward**. To visualize this westward movement of patterns, note that the geostrophic flow v advects low PV from the south and high PV from the north. The initial perturbation can be caused by a local convergence of surface flow that pushes the interface downward.



Baroclinic Instability

- Two-layer QG system



Consider a two-layer system with $H_1 = H_2 = H$. The interface $\bar{\xi}$ is tilted, and is associated with a background flow (in x-direction) under geostrophic balance, given as

$$\overline{\psi}_1 = -Uy, \qquad \overline{\psi}_2 = Uy, \qquad \bar{\xi} = \frac{f_o}{g'}\left(\overline{\psi}_2 - \overline{\psi}_1\right) = \frac{2f_oU}{g'}y$$

Assume $|\bar{\xi}| \ll H$ with the β-plane $f = f_o + \beta y$. The background PV fields are simplified as

$$\bar{q}_1 = \frac{f}{H-\bar{\xi}} \approx \frac{f_o + \beta y}{H} + \frac{f_o\bar{\xi}}{H^2}, \qquad \bar{q}_2 = \frac{f}{H+\bar{\xi}} \approx \frac{f_o + \beta y}{H} - \frac{f_o\bar{\xi}}{H^2}$$

The gradient of background PV becomes

$$\nabla\bar{q}_1 = \frac{\beta}{H}\hat{\mathbf{j}} + \frac{f_o}{H^2}\nabla\bar{\xi} = \frac{\beta+\hat{\beta}}{H}\hat{\mathbf{j}}, \qquad \nabla\bar{q}_2 = \frac{\beta-\hat{\beta}}{H}\hat{\mathbf{j}}, \qquad \hat{\beta} = \frac{2U}{\left(L_r^{\mathrm{bc}}\right)^2}$$

The perturbed fields are denoted as $\xi, \boldsymbol{u}_i, \psi_i$. With the similar assumptions, the contribution to PV anomaly from the free surface η can be neglected, which leads to

$$q_1' \approx \frac{1}{H}\left[\nabla^2\psi_1 + \frac{\psi_2 - \psi_1}{\left(L_r^{\mathrm{bc}}\right)^2}\right], \qquad q_2' = \frac{1}{H}\left[\nabla^2\psi_2 - \frac{\psi_2 - \psi_1}{\left(L_r^{\mathrm{bc}}\right)^2}\right]$$

The contribution from η is neglected since $L \ll L_r$. The conservation of total PV gives

$$\frac{\mathrm{D}_1 q_1'}{\mathrm{D}t} = \frac{\partial q_1'}{\partial t} + J(\psi_1, q_1') + U\frac{\partial q_1'}{\partial x} = -v_1\frac{\partial \bar{q}_1}{\partial y}$$
$$\frac{\mathrm{D}_2 q_2'}{\mathrm{D}t} = \frac{\partial q_2'}{\partial t} + J(\psi_2, q_2') - U\frac{\partial q_2'}{\partial x} = -v_2\frac{\partial \bar{q}_2}{\partial y}$$

Now the advection also includes the background flow. The linearized equations become

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla^2\psi_1 + \frac{\psi_2 - \psi_1}{\left(L_r^{\mathrm{bc}}\right)^2}\right] + \left(\beta + \hat{\beta}\right)\frac{\partial \psi_1}{\partial x} = 0$$

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\left[\nabla^2\psi_2 - \frac{\psi_2 - \psi_1}{\left(L_r^{\mathrm{bc}}\right)^2}\right] + \left(\beta - \hat{\beta}\right)\frac{\partial \psi_2}{\partial x} = 0$$

Similarly using the barotropic and baroclinic modes, we have

$$\frac{\partial}{\partial t}(\nabla^2\psi_{\rm bt}) + \beta\frac{\partial\psi_{\rm bt}}{\partial x} + U\frac{\partial}{\partial x}(\nabla^2\psi_{\rm bc}) = 0$$

$$\frac{\partial}{\partial t}\left[\nabla^2\psi_{\rm bc} - \frac{2\psi_{\rm bc}}{\left(L_r^{\rm bc}\right)^2}\right] + \beta\frac{\partial\psi_{\rm bc}}{\partial x} + U\frac{\partial}{\partial x}(\nabla^2\psi_{\rm bt}) + \hat{\beta}\frac{\partial\psi_{\rm bt}}{\partial x} = 0$$

With the background baroclinic current, the equations are coupled through:

- Advection of background PV gradient from the tilted interface by the barotropic mode
- Advection of barotropic / baroclinic vorticity by the background baroclinic flow

Coupling terms

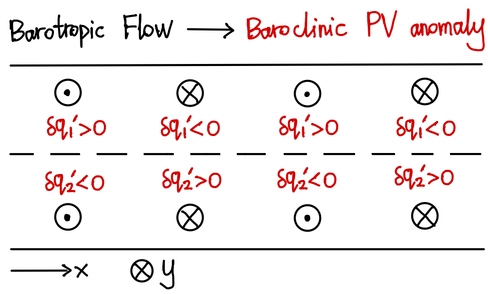
The advection of background PV gradient by the barotropic mode is

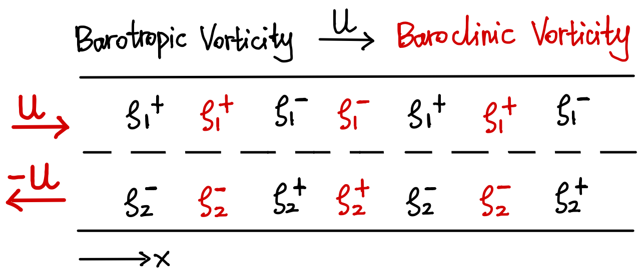
$$\hat{\beta}\frac{\partial\psi_{\rm bt}}{\partial x} = \hat{\beta}v_{\rm bt} = Hv_{\rm bt}\frac{\partial\bar{q}_1}{\partial y} = -Hv_{\rm bt}\frac{\partial\bar{q}_2}{\partial y}$$

The advection of barotropic / baroclinic vorticity by the background baroclinic flow is

$$U\frac{\partial}{\partial x}(\nabla^2\psi_{\rm bc}) \qquad U\frac{\partial}{\partial x}(\nabla^2\psi_{\rm bt})$$

These coupling terms are illustrated below. The barotropic flow advects background PV field and generates a baroclinic PV anomaly. The background flow advects the vorticity field of one mode and generates the other mode.





➢ Baroclinic instability

Now consider the f-plane with $\beta = 0$. For linear stability analysis, we seek wave solution and this leads to the eigenvalue problem

$$\omega\hat{\psi}_{\rm bt} - Uk\hat{\psi}_{\rm bc} = 0, \qquad \omega\left[K^2 + \frac{2}{\left(L_r^{\rm bc}\right)^2}\right]\hat{\psi}_{\rm bc} - UkK^2\hat{\psi}_{\rm bt} + k\hat{\beta}\hat{\psi}_{\rm bt} = 0$$

The eigenvalues and eigenvectors are solved as

$$\omega^2 = k^2U^2\cdot\frac{K^2 - 2/\left(L_r^{\rm bc}\right)^2}{K^2 + 2/\left(L_r^{\rm bc}\right)^2}, \qquad \hat{\psi}_{\rm bc} = \frac{\omega}{kU}\hat{\psi}_{\rm bt}, \qquad K^2 = k^2 + l^2$$

When $K^2 < 2/\left(L_r^{\mathrm{bc}}\right)^2$, we have growing modes with imaginary frequency

$$\omega = i\omega_I = ikU\gamma, \qquad \gamma = \left[\frac{2/\left(L_r^{\mathrm{bc}}\right)^2 - K^2}{K^2 + 2/\left(L_r^{\mathrm{bc}}\right)^2}\right]^{1/2}$$

The fastest growing mode occurs at $l = 0$ with streamwise wavelength

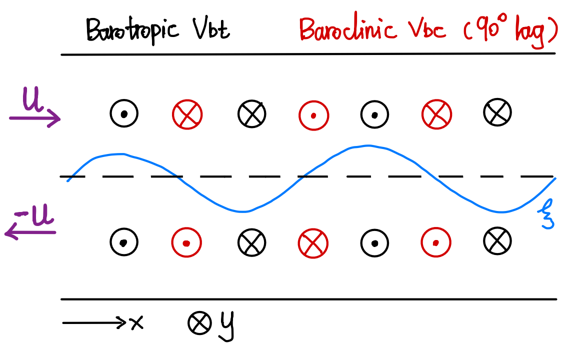
$$k \approx \frac{0.91}{L_r^{\mathrm{bc}}}, \qquad \lambda_x = \frac{2\pi}{k} \approx 6.7\, L_r^{\mathrm{bc}} \approx 67\ \mathrm{km}$$

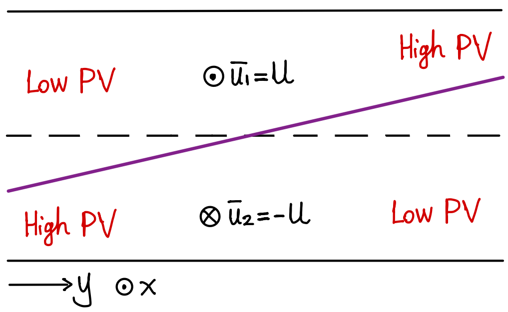
For unstable modes, the baroclinic $\hat{\psi}_{\mathrm{bc}}$ lags the barotropic $\hat{\psi}_{\mathrm{bt}}$ by 90°. This is essential for the instability mechanism.

$$\psi_{\mathrm{bt}} = \hat{\psi}_{\mathrm{bt}}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \qquad \psi_{\mathrm{bc}} = i\gamma\hat{\psi}_{\mathrm{bt}}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} = \gamma\hat{\psi}_{\mathrm{bt}}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t + \pi/2)}$$

Instability mechanism

The structure of the fastest growing mode indicates the phase lag between the two modes.

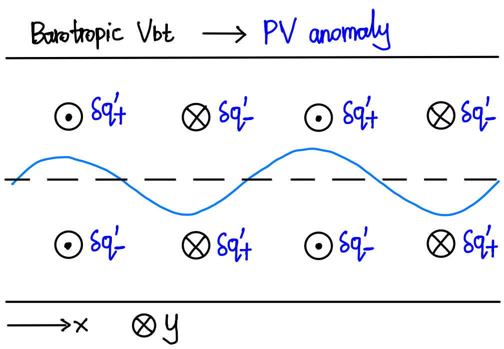


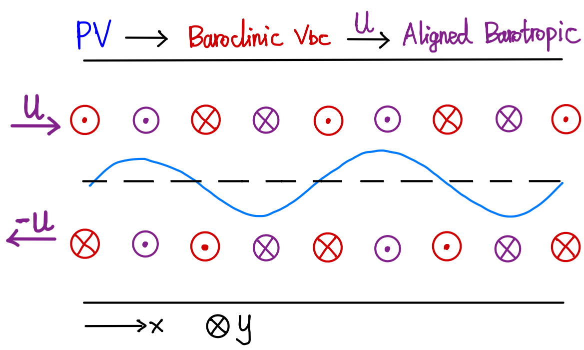


The instability mechanism can be summarized into two stages:

Stage 1. Barotropic perturbation advects background PV, setting up baroclinic vorticity

Stage 2. Background flow advects baroclinic vorticity, reinforcing barotropic perturbation





Influence of planetary vorticity

When $\beta > \hat{\beta}$, the PV gradient in both layers have the same sign. Then, a barotropic perturbation will not create a baroclinic PV anomaly, and thus no growth. An essential criterion for mode growth is that the PV gradients have opposite signs in the two layers.

- Energetics of baroclinic instability

Baroclinic instability derives its KE from the release of APE through driving a net overturning circulation that flattens out the tilted interface. Averaged over the x-direction (N-S), the eddies drive a net northward transport in the upper layer, and southward transport in the lower layer. However, the averaged N-S flow is zero. To see this, we have

$$v_1 = \frac{1}{2}\left(\frac{\partial \psi_{\mathrm{bt}}}{\partial x} + \frac{\partial \psi_{\mathrm{bc}}}{\partial x}\right), \qquad v_2 = \frac{1}{2}\left(\frac{\partial \psi_{\mathrm{bt}}}{\partial x} - \frac{\partial \psi_{\mathrm{bc}}}{\partial x}\right)$$

With wave solution, their average over one wavelength is zero. But the release of APE is in fact governed by the net volume transport in layers, instead of the averaged velocity. The transport also depends on the layer thickness which is modified by the instability. Note that

$$h_1 = H - \bar{\xi} - \xi', \qquad h_2 = H + \bar{\xi} + \xi', \qquad \xi = \frac{f_o}{g'}(\psi_2 - \psi_1) = -\frac{2f_o}{g'}\psi_{\mathrm{bc}}$$

The net N-S transport in the upper layer is

$$V_1 = \frac{1}{\lambda_x}\int_0^{\lambda_x} v_1 h_1 \, \mathrm{d}x = -\frac{1}{\lambda_x}\int_0^{\lambda_x} v_1 \xi' \, \mathrm{d}x = \frac{f_o}{g'\lambda_x}\int_0^{\lambda_x}\left(\frac{\partial \psi_{\mathrm{bt}}}{\partial x} + \frac{\partial \psi_{\mathrm{bc}}}{\partial x}\right)\psi_{\mathrm{bc}} \, \mathrm{d}x$$

Consider the solution for a growing mode, which is

$$\psi_{\mathrm{bt}} = \psi_o e^{\omega_I t}\cos(kx), \qquad \psi_{\mathrm{bc}} = \mathrm{Re}\left\{i\gamma\psi_o e^{\omega_I t} e^{ikx}\right\} = -\gamma\psi_o e^{\omega_I t}\sin(kx)$$

It can be shown that the net N-S transport is positive

$$V_1 = \frac{f_o}{g'\lambda_x} k\gamma\psi_o^2 e^{2\omega_I t}\int_0^{\lambda_x}[\sin(kx) + \gamma\cos(kx)]\sin(kx)\, \mathrm{d}x = \frac{f_o}{2g'}k\gamma\psi_o^2 e^{2\omega_I t}$$

From the result, we notice that

$$V_1 \propto -\int_0^{\lambda_x} \xi' v_{\mathrm{bt}} \, \mathrm{d}x$$

The transport depends on the correlation between the barotropic N-S velocity and the interface perturbation. In the upper layer, they are negatively correlated, which indicates a net northward transport. Vice versa, there is a net southward transport in the lower layer with $V_2 = -V_1$. These net transports correspond to a net release of APE.

Frictional Effects

- Ekman layer

Frictional force from turbulent momentum flux

The effects of friction are important in boundary layers near the surface or bottom of the ocean. The flows in the Ekman layers, for example, affect the dynamics of the entire water column. Friction in the boundary layer is associated with turbulent processes that transport momentum flux down the gradient. The vertical flux of horizontal momentum by turbulence is proportional to the vertical shear of mean horizontal flow through **eddy viscosity** ν_e, which is

$$\overline{u'w'} = -\nu_e \frac{\partial \bar{u}}{\partial z}, \qquad \overline{v'w'} = -\nu_e \frac{\partial \bar{v}}{\partial z}$$

The lateral friction can be neglected. In practice, ν_e is not constant and varies with properties of the mean flow, surface forcing, roughness of the bathymetry, etc., and has to be parameterized in ocean circulation models. The typical range of ν_e is

$$10^{-5}\ \mathrm{m^2/s} < \nu_e < 0.1\ \mathrm{m^2/s}$$

The frictional force (per unit mass) generated by these turbulent momentum fluxes is

$$\boldsymbol{F} = -\frac{\partial}{\partial z}\left(-\nu_e \frac{\partial \bar{\boldsymbol{u}}}{\partial z}\right) = \nu_e \frac{\partial^2 \bar{\boldsymbol{u}}}{\partial z^2}$$

Ekman number

For simplicity, consider a constant ν_e. The Boussinesq equation becomes

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho_o}\nabla p + b\hat{\mathbf{k}} + \nu_e \frac{\partial^2 \bar{\boldsymbol{u}}}{\partial z^2}$$

Assume the fluid is homogeneous with $\rho = \rho_o$ and $b = 0$. The non-dimensional equation is

$$\mathrm{Ro_t}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \mathrm{Ro}\left(\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{u}\right) - \tilde{v} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mathrm{Ek}\,\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}$$

$$\mathrm{Ro_t}\frac{\partial \tilde{v}}{\partial \tilde{t}} + \mathrm{Ro}\left(\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{v}\right) + \tilde{u} = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \mathrm{Ek}\,\frac{\partial^2 \tilde{v}}{\partial \tilde{z}^2}$$

$$\mathrm{Ro_t}\,\Gamma^2\frac{\partial \tilde{w}}{\partial \tilde{t}} + \mathrm{Ro}\,\Gamma^2\left(\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{w}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{z}} + \mathrm{Ek}\,\Gamma^2\frac{\partial^2 \tilde{w}}{\partial \tilde{z}^2}$$

The key non-dimensional parameters are

$$\mathrm{Ro_t} = \frac{1}{fT}, \qquad \mathrm{Ro} = \frac{U}{fL}, \qquad \Gamma = \frac{H}{L}, \qquad \mathrm{Ek} = \frac{\nu_e}{fH^2}$$

The Ekman number compares the frictional force to the Coriolis force. In the ocean, we have

$$\mathrm{Ek} = \frac{\nu_e}{fH^2} \sim \frac{10^{-3}\ \mathrm{m^2/s}}{10^{-4}\ \mathrm{s^{-1}} \times 10^6\ \mathrm{m^2}} \sim 10^{-5} \ll 1$$

When all these parameters are small, the governing equations describe the geostrophic flow invariant over depth. However, the solution cannot satisfy the no-slip boundary condition at the bottom. This implies that a thin boundary layer must develop where the frictional force is as important as the Coriolis force, and locally the Ekman number is of $O(1)$.

Ekman boundary layer

We decompose the balanced flow into the interior flow (geostrophic and ageostrophic) and the Ekman flow as

$$u = u_i + u_i^{\mathrm{ag}} + u_e, \qquad v = v_i + v_i^{\mathrm{ag}} + v_e$$

The Ekman flow is confined within the vertical scale δ of the Ekman layer

$$\mathrm{Ek} = \frac{\nu_e}{f\delta^2} \sim 1, \qquad \delta \sim \sqrt{\nu_e/f}$$

Away from the boundary, we have

$$u_e, v_e \to 0, \qquad u, v \to u_i, v_i, \qquad \text{for } z \gg \delta$$

The geostrophic interior flow (not varying with depth) is given by

$$-f v_i = -\frac{1}{\rho_o}\frac{\partial p}{\partial x}, \qquad f u_i = -\frac{1}{\rho_o}\frac{\partial p}{\partial y}$$

The ageostrophic component is of order $O\left(\sqrt{\mathrm{Ek}}\right)$ and neglected for the leading order balance, but it is important for the evolution of Ekman flow. Within the Ekman layer, we have

$$-f(v_i + v_e) = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \nu_e \frac{\partial^2 u_e}{\partial z^2}, \qquad f(u_i + u_e) = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + \nu_e \frac{\partial^2 v_e}{\partial z^2}$$

Since pressure does not vary through the Ekman layer, subtracting the equations gives

$$-f v_e = \nu_e \frac{\partial^2 u_e}{\partial z^2}, \qquad f u_e = \nu_e \frac{\partial^2 v_e}{\partial z^2}$$

This is the Ekman balance. The boundary conditions are specified as

$$u \sim u_i + u_e = 0, \qquad v \sim v_i + v_e = 0, \qquad \text{at } \mathrm{z} = 0$$

$$u_e \to 0, \qquad v_e \to 0, \qquad \text{at } z \to \infty$$

We can obtain a single governing equation for u_e as

$$u_e = \frac{\nu_e}{f}\frac{\partial^2 v_e}{\partial z^2} = -\left(\frac{\nu_e}{f}\right)^2 \frac{\partial^4 v_e}{\partial z^4}, \qquad \frac{\partial^4 v_e}{\partial z^4} + \frac{f^2}{\nu_e^2} u_e = 0$$

We seek solution of the form $e^{\lambda z}$ with Re $\lambda < 0$, which leads to

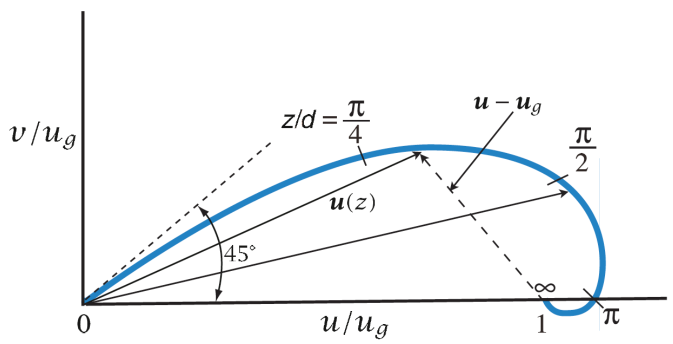
$$\lambda^4 + \frac{f^2}{\nu_e^2} = 0, \qquad \lambda = \frac{1}{\delta_e}(-1 \pm i), \qquad \delta_e = \sqrt{\frac{2\nu_e}{f}} = \sqrt{2}\cdot\sqrt{\mathrm{Ek}}\cdot H$$

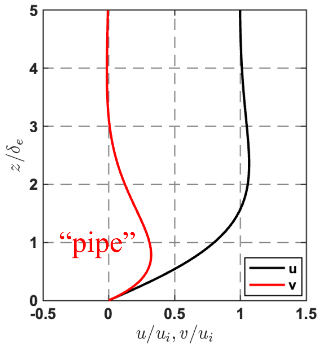
The length scale δ_e is the Ekman layer depth. With the boundary conditions at $z = 0$, we have

$$u_e = -v_i\, e^{-z/\delta_e}\sin(z/\delta_e) - u_i\, e^{-z/\delta_e}\cos(z/\delta_e)$$

$$v_e = u_i\, e^{-z/\delta_e}\sin(z/\delta_e) - v_i\, e^{-z/\delta_e}\cos(z/\delta_e)$$

The Ekman flow spirals with depth **to the left** (Northern Hemisphere) of the geostrophic flow.





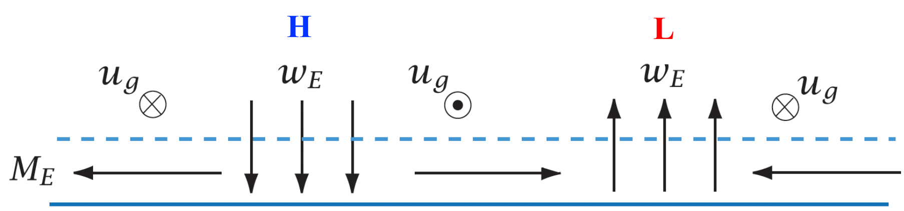
To understand this, consider a geostrophic flow with $v_i = 0$. In the Ekman layer, PGF is not entirely balanced by the Coriolis force, as the Ekman flow u_e reduces the total zonal flow to satisfy the no-slip bottom boundary condition. This unbalanced PGF would drive a flow in the $+y$-direction **to the left** of u_i. This tendency is eventually counteracted by the friction.

$$-\frac{1}{\rho_o}\frac{\partial p}{\partial y} - f(u_i + u_e) + \nu_e \frac{\partial^2 v_e}{\partial z^2} = 0$$

The Ekman flow is analogous to a pipe flow **down pressure gradient**. In this case, the N-S flow v_e is governed by the balance between PGF and friction, and its profile is nearly parabolic with negative curvature. The Ekman layer breaks the rotational constraint set by geostrophy.

➢ Spin-down of geostrophic flow by Ekman transport

Flows in the Ekman layer transport fluid down pressure gradient. Consider a cyclone setup, the bottom Ekman flows are convergent and fill in the low pressure region. This process weakens the PGF and thus the geostrophic flow.



Ekman transport

The Ekman transport is defined as

$$M_e^x = \int_0^{z_T} u_e \, \mathrm{d}z\,, \qquad M_e^y = \int_0^{z_T} v_e \, \mathrm{d}z$$

The upper bound $z_T > \delta_e$ is the height where $u_e, v_e \approx 0$. The turbulent stress is given as

$$\tau_x = \rho_o \nu_e \frac{\partial u_e}{\partial z}, \qquad \tau_y = \rho_o \nu_e \frac{\partial v_e}{\partial z}$$

Using the Ekman balance, we can evaluate the integrals as

$$M_e^x = \frac{\nu_e}{\rho_o f}\int_0^{z_T} \frac{\partial \tau_y}{\partial z}\, \mathrm{d}z = -\frac{\tau_b^y}{\rho_o f}, \qquad M_e^y = \frac{\tau_b^x}{\rho_o f}, \qquad \mathbf{M}_\mathrm{e} = \frac{\hat{\mathbf{k}} \times \boldsymbol{\tau}_b}{\rho_o f}$$

The Ekman spiral solution evaluates the bottom stress as

$$\tau_b^x = \rho_o \nu_e \frac{\partial u_e}{\partial z}\bigg|_{z=0} = \frac{\rho_o \nu_e}{\delta_e}(u_i - v_i), \qquad \tau_b^y = \rho_o \nu_e \frac{\partial v_e}{\partial z}\bigg|_{z=0} = \frac{\rho_o \nu_e}{\delta_e}(u_i + v_i)$$

Therefore, the Ekman transport becomes

$$\mathbf{M}_\mathrm{e} = \frac{\delta_e}{2}[(u_i - v_i)\,\hat{\mathbf{j}} - (u_i + v_i)\,\hat{\mathbf{i}}]$$

It can also be related to PGF by using the interior geostrophic flow, which leads to

$$\mathbf{M}_\mathrm{e} = \frac{\delta_e}{2f}\left(-\frac{1}{\rho_o}\nabla_h p + \frac{1}{\rho_o}\nabla_h p \times \hat{\mathbf{k}}\right)$$

This implies that Ekman transport has a component down pressure gradient. For a low pressure center, the bottom Ekman transport is convergent and generates an upward motion.

Ekman pumping / suction

We similarly decompose $w = w_i + w_e$. Note that the geostrophic flow does not have a vertical component, and w_i here is purely ageostrophic. Due to different vertical scales of the interior (ageostrophic) and Ekman flows, the continuity should be considered separately as

$$\frac{\partial u_i^{\mathrm{ag}}}{\partial x} + \frac{\partial v_i^{\mathrm{ag}}}{\partial y} + \frac{\partial w_i^{\mathrm{ag}}}{\partial z} = 0, \qquad \frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y} + \frac{\partial w_e}{\partial z} = 0$$

Since there is no normal flow at the bottom, we have

$$w = w_i^{\mathrm{ag}} + w_e = 0, \qquad \text{at } \mathrm{z} = 0$$

Integrating the continuity equation gives

$$\int_0^{z_T} \frac{\partial w_e}{\partial z}\,\mathrm{d}z = -w_e|_{z=0} = -\int_0^{z_T}\left(\frac{\partial u_e}{\partial x} + \frac{\partial v_e}{\partial y}\right)\mathrm{d}z = -\nabla_h \cdot \mathbf{M}_\mathrm{e}$$

The Ekman pumping / suction is reflected by the interior flow w_i at the boundary, which is

$$w_i^{\mathrm{ag}}\big|_{z=0} = -\nabla_h \cdot \mathbf{M}_\mathrm{e}$$

Positive $w_i|_{z=0} > 0$ implies Ekman pumping of fluid from the bottom Ekman layer into the interior. For surface Ekman layer, the terms are flipped. The convergence of Ekman transport can be evaluated as

$$-\nabla_h \cdot \mathbf{M}_\mathrm{e} = \frac{\delta_e}{2}\left[\left(\frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y}\right) + \left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right)\right] = \frac{\delta_e}{2}\zeta_i$$

We thus obtain the relation between the Ekman pumping and the vorticity

$$w_i^{\mathrm{ag}}\big|_{z=0} = \frac{\delta_e}{2}\zeta_i$$

As a summary, Ekman transport tends to fill in low pressure regions, and remove mass from the high pressure regions.

Spin-down of geostrophic flow

Consider a geostrophic flow between two solid boundaries at $z = 0$ and $z = H$. For the top and bottom Ekman layers, we have

$$w_i|_{z=H} = -\frac{\delta_e}{2}\zeta_i, \qquad w_i|_{z=0} = \frac{\delta_e}{2}\zeta_i$$

Based on the vortex dynamics, we have

$$\frac{\partial \zeta_i}{\partial t} = f\frac{\partial w_i}{\partial z}$$

The vorticity evolves by vortex stretching / squashing of planetary vorticity. For a cyclone, the geostrophic flow has $\zeta_i > 0$, while the Ekman flow induces a vortex squashing that reduces the magnitude of ζ_i. For an anticyclone, the geostrophic flow has $\zeta_i < 0$, while the Ekman flow induces a vortex stretching that also reduces the magnitude of ζ_i.

To solve the vorticity equation, note that the barotropic (ageostrophic) flow implies constant ζ_i over depth. The continuity also implies that $\partial w_i/\partial z$ is also constant over depth, which gives

$$\frac{\partial w_i}{\partial z} = \frac{w_i|_{z=H} - w_i|_{z=0}}{H} = -\frac{\delta_e}{H}\zeta_i$$

From continuity of ageostrophic component, $U_{\text{ag}} \sim \sqrt{\text{Ek}}\, U_g$. The vorticity equation becomes

$$\frac{\partial \zeta_i}{\partial t} = -\frac{\delta_e}{H}f\zeta_i, \qquad \zeta_i = \zeta_o \exp\left(-\frac{\delta_e}{H}ft\right)$$

The e-folding time for vorticity spin-down is

$$\tau_{\text{sd}} = \frac{H}{\delta_e}f^{-1}$$

Note that the Ekman number describes the ratio between the vertical scales, we have

$$\text{Ek} = \frac{\nu_e}{fH^2} = \frac{1}{2}\left(\frac{\delta_e}{H}\right)^2, \qquad \tau_{\text{sd}} = \frac{H}{\delta_e}f^{-1} = \frac{1}{\sqrt{2}}\text{Ek}^{-1/2}f^{-1} = \frac{1}{2\sqrt{2}\pi}\text{Ek}^{-1/2}T_i$$

For small Ekman number, the spin-down time is much longer than the inertial period.

Effects of rotation

The frictional force does not directly decelerate the geostrophic flow. It is the vortex stretching or squashing of planetary vorticity that spins down the geostrophic flow. In fact, the rotation speeds up the spin-down process relative to a non-rotating fluid. Without rotation, the spin-down timescale is governed by the diffusive time, which is much longer as shown below

$$\tau_{\text{sd}}^{\text{NR}} = \frac{H^2}{\nu_e} = \text{Ek}^{-1}f^{-1}, \qquad \frac{\tau_{\text{sd}}^{\text{NR}}}{\tau_{\text{sd}}} \sim \sqrt{2/\text{Ek}} \gg 1$$

Two-dimensional Turbulence

Consider a two-dimensional flow invariant in z-direction with no vertical velocity $w = 0$. The continuity equation implies that we can describe the flow with a stream function ψ

$$\nabla \cdot \boldsymbol{u} = \nabla_h \cdot \boldsymbol{u} = 0, \qquad u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

Physically this could represent a geostrophic flow, as the vertical rigidity imposed by rotation makes the flow nearly invariant in z-direction, as shown by the Taylor-Proudman theorem

$$\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = \boldsymbol{0}, \qquad \boldsymbol{\Omega} = \Omega\, \hat{\mathbf{k}} \qquad \Rightarrow \qquad \frac{\partial \boldsymbol{u}}{\partial z} = \boldsymbol{0}$$

For inviscid flow, the momentum equations are

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho_o}\frac{\partial p}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y}$$

The equation for KE can be obtained as

$$\frac{\partial \mathrm{KE}}{\partial t} = -\nabla_h \cdot \left(\boldsymbol{u}\, \mathrm{KE} + \frac{p\boldsymbol{u}}{\rho_o}\right) = -\nabla_h \cdot \boldsymbol{F}_E, \qquad \mathrm{KE} = \frac{1}{2}(u^2 + v^2)$$

For a volume V, if the energy flux $\boldsymbol{F}_E$ is either zero or periodic at the boundaries, we have

$$\frac{\partial}{\partial t}\int_V \mathrm{KE}\, \mathrm{d}V = -\int_{\partial V} \boldsymbol{F}_E \cdot \hat{\boldsymbol{n}}\, \mathrm{d}A = 0$$

In the Fourier domain, the KE spectrum is defined as

$$\widehat{\mathrm{KE}} = \frac{1}{2}(\hat{u}\hat{u}^* + \hat{v}\hat{v}^*), \qquad u = \int \hat{u}(\boldsymbol{k}, t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}^3\boldsymbol{k}$$

From the Parseval theorem, we have

$$\int_V \mathrm{KE}\, \mathrm{d}V = \int_{\hat{V}} \widehat{\mathrm{KE}}\, \mathrm{d}^3\boldsymbol{k}, \qquad \frac{\partial}{\partial t}\int_{\hat{V}} \widehat{\mathrm{KE}}\, \mathrm{d}^3\boldsymbol{k} = 0$$

Now we focus on the vertical vorticity component ζ and its governing equation. Since $w = 0$, the vorticity is advected like a tracer

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi, \qquad \frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0$$

We can define the enstrophy $\mathcal{E}$ and obtain its governing equation as

$$\mathcal{E} = \frac{1}{2}\zeta^2, \qquad \frac{\partial \mathcal{E}}{\partial t} = -\nabla_h \cdot (\boldsymbol{u}\mathcal{E}) = -\nabla_h \cdot \boldsymbol{F}_\mathcal{E}$$

If the enstrophy flux $\boldsymbol{F}_\mathcal{E}$ is either zero or periodic at the boundaries, we similarly have

$$\frac{\partial}{\partial t}\int_V \mathcal{E}\, \mathrm{d}V = \frac{\partial}{\partial t}\int_{\hat{V}} \hat{\mathcal{E}}\, \mathrm{d}^3\boldsymbol{k} = 0, \qquad \hat{\mathcal{E}} = \frac{1}{2}\hat{\zeta}\hat{\zeta}^*$$

The enstrophy spectrum is related to the KE spectrum. Note that

$$\widehat{\mathrm{KE}} = \frac{1}{2}(\hat{u}\hat{u}^* + \hat{v}\hat{v}^*) = \frac{1}{2}(k^2 + l^2)\,\hat{\psi}\hat{\psi}^*, \qquad \hat{\zeta} = -(k^2 + l^2)\,\hat{\psi}$$

The enstrophy spectrum then can be written as

$$\hat{\mathcal{E}} = \frac{1}{2}\hat{\zeta}\hat{\zeta}^* = \frac{1}{2}K^4\hat{\psi}\hat{\psi}^* = K^2\widehat{\mathrm{KE}}, \qquad K^2 = k^2 + l^2$$

The spectrum can be quantified based on the centroid wavenumber K_E and its bandwidth ΔK_E

$$K_E = \frac{\int K \cdot \widehat{\mathrm{KE}}\, \mathrm{d}K}{\int \widehat{\mathrm{KE}}\, \mathrm{d}K}, \qquad \Delta K_E = \frac{\int (K - K_E)^2\, \widehat{\mathrm{KE}}\, \mathrm{d}K}{\int \widehat{\mathrm{KE}}\, \mathrm{d}K}$$

We want to know how K_E evolves with time, which implies the evolution of spatial scale of 2D turbulence, either forward cascade (K_E increases with time) or inverse cascade (K_E decreases with time). A general property of turbulent flows is that they distribute energy to different wavenumber components, and thus increase the bandwidth with $\partial(\Delta K_E)/\partial t > 0$.

From the definition of ΔK_E, we have

$$\Delta K_E = \frac{\int \hat{\mathcal{E}}\, \mathrm{d}K}{\int \widehat{\mathrm{KE}}\, \mathrm{d}K} - K_E^2, \qquad \frac{\partial(\Delta K_E)}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\int \hat{\mathcal{E}}\, \mathrm{d}K}{\int \widehat{\mathrm{KE}}\, \mathrm{d}K}\right] - 2K_E\frac{\partial K_E}{\partial t}$$

Under the assumption of conservation of $\hat{\mathcal{E}}$ and $\widehat{\mathrm{KE}}$ over the total volume V, we obtain

$$\frac{\partial(\Delta K_E)}{\partial t} = -2K_E\frac{\partial K_E}{\partial t} > 0, \qquad \frac{\partial K_E}{\partial t} < 0$$

It implies that the centroid wavenumber K_E becomes smaller, moves to larger scales and thus there is an **inverse cascade**. In turbulent flows with low Rossby number, the kinetic energy is transferred from small to large scales, following an inverse cascade. As the eddies grow in size, they extend deeper in the water column and become more barotropic.

Turbulent motions strongly constrained by Earth's rotation follow the rules of 2D turbulence:

- Vortices that spin in the same direction orbit one another (Fujiwara effect).
- Vortices merge and become larger in size.