Asymptotic Analysis of Differential Equations (1): Linear ODE

We analyze the asymptotic expansion of the solution for linear ODE on the complex plane. Denote the domain $\Omega \subseteq \mathbb{C}$, the meromorphic function domain $m(\Omega)$ and holomorphic function ring $\mathbb{O}(\Omega)$, the $n \times n$ matrix $M(m(\Omega), n)$ with its elements composed of function $f \in m(\Omega)$. For a matrix $A \in M(m(\Omega), n)$, a linear ODE can be represented by

$$y'(z) = A(z)y(z), \qquad y \in \mathbb{C}^n$$

Based on variation of parameters, we only need to study the homogeneous equation. For an initial value problem $y(z_0) = y_0$, we can first solve for the matrix equation

$$Y'(z) = A(z)Y(z), \qquad Y(z_0) = I \qquad (*)$$

With this fundamental matrix, we have $y = Yy_0$. Hence, we will focus on this matrix equation.

- Qualitative theory of solutions (6.1)

Cauchy Theorem

For $y' = f(z, y)$ with $y(z_0) = y_0$, if f is analytic then there exists a unique analytic solution.

Consider $(*)$ has a solution Y_0 around z_0. For a path γ starting from z_0, we can perform analytic continuation of Y_0 into a neighborhood of γ. When γ goes back to z_0, we can obtain another solution Y_γ. Then we state that there exists an invertible $C_\gamma \in GL(\mathbb{C}, n)$ such that $Y_\gamma = Y_0 C_\gamma$. We define a mapping $\rho_A: \gamma \mapsto C_\gamma$, and when γ_1 and γ_2 are homotopic, we have $C_{\gamma_1} = C_{\gamma_2}$. This implies that $\rho_A: \pi_1(\Omega^*, z_0) \to GL(\mathbb{C}, n)$, with Ω^* is the domain with poles removed. Moreover, if $\gamma = \gamma_1 \circ \gamma_2$, then $C_\gamma = C_{\gamma_1} C_{\gamma_2}$. So ρ_A is a group homomorphism and a representation of π_1.

For the matrix equation $(*)$, consider a transformation $P \in GL(\mathbb{O}(\Omega), n)$ and denote $Z = PY$.

$$Z' = P'Y + PY' = (P'P^{-1} + PAP^{-1})Z = BZ$$

The two mappings ρ_A and ρ_B are equivalent. $P'P^{-1} + PAP^{-1}$ is a meromorphic connection on vector bundles on a complex manifold, an example of Riemann-Hilbert correspondence.

Local problem

Let z_0 is a pole of A, with r denoted as the Poincaré rank. This implies that

$$A(z) = (z - z_0)^{-r} \tilde{A}(z), \qquad \tilde{A}(z) \in M(\mathbb{O}(z_0), n), \qquad \tilde{A}(z_0) \neq 0$$

Without loss of generality, take $z_0 = 0$ and we have

$$z^r Y'(z) = \tilde{A}(z)Y(z)$$

Since z_0 is a pole, $Y(z_0)$ may not exists, and we only focus on the equation. The solution is highly influenced by the Poincaré rank r.

When $r=1$, consider a constant matrix A and we have

$$zY'(z)=AY(z)$$

Select a branch cut C from $z=0$, we have $\ln z\in\mathbb{O}(\Omega\setminus C)$ and $Y(z)=e^{A\ln z}$. Consider A has the Jordan normal form $A=PJP^{-1}$ with $J=\Lambda+N$. Then we can write

$$Y(z)=P\left(e^{J\ln z}\right)P^{-1},\qquad e^{J\ln z}=\Lambda^z\left(\sum_{k=0}^{n}\frac{\ln^k z}{k!}N^k\right)$$

The singularity is regular for $r=1$. Going around $z=0$, we obtain C_γ as follows

$$Y_0\left(ze^{2\pi i}\right)=e^{A(\ln z+2\pi i)}=Y_0(z)e^{2\pi iA},\qquad C_\gamma=e^{2\pi iA}$$

When $r=2$, still consider a constant matrix A and we have

$$z^2Y'(z)=AY(z),\qquad Y(z)=e^{-A/z}$$

Now $z=0$ becomes an essential singularity, and the solution only exists in a sector. We cannot go around $z=0$ as in the previous case. For $r\geq 2$, the singularity is irregular.

➢ Majorant series & Cauchy theorem

Let $\Omega\subseteq\mathbb{C}^{d+1}$ with coordinates $\tilde{y}=(z,y)$ with $z\in\mathbb{C}$, $y\in\mathbb{C}^d$ and a function $f\in\mathbb{O}(\Omega,\mathbb{C}^d)$.

$$y'=f(z,y),\qquad \tilde{y}_0=(z_0,y_0)\in\Omega$$

The function f is analytic when there exists $r>0$ such that when $\tilde{y}\in B(\tilde{y}_0,r)$, the following series is convergent

$$f(\tilde{y})=\sum_{j\geq 0}c_j(\tilde{y}-\tilde{y}_0)^j,\qquad j=\{j_0,j_1,\cdots,j_d\}$$

The neighborhood is $B(\tilde{y}_0,r)=\{\tilde{y}\in\Omega\mid|\tilde{y}-\tilde{y}_0|<r\}$ with the L_∞ norm $|\tilde{y}|=\max|y_i|$. The above notation means

$$f(\tilde{y})=\sum_{j_0,\cdots,j_d\geq 0}c_{j_0\cdots j_d}(z-z_0)^{j_0}(y_1-y_{10})^{j_1}\cdots(y_d-y_{d0})^{j_d}$$

Majorant series

Consider a formal power series $f(\tilde{y})$. If there exists another series $F(\tilde{y})$ such that $\forall j$ we have $|a_j|\leq A_j$, then $F(\tilde{y})$ is a majorant series of $f(\tilde{y})$.

$$f(\tilde{y})=\sum_{j\geq 0}a_j(\tilde{y}-\tilde{y}_0)^j,\qquad F(\tilde{y})=\sum_{j\geq 0}A_j(\tilde{y}-\tilde{y}_0)^j$$

If $F(\tilde{y})$ converges in $B(\tilde{y}_0,r)$, then $f(\tilde{y})$ also converges. We can then call $F(\tilde{y})$ as the majorant function of $f(\tilde{y})$.

Corollary. If $f(\tilde{y})$ is analytic around $\tilde{y}_0$, i.e., it can be expanded on $B(\tilde{y}_0, R)$ into a convergent series, then for any $r \in (0, R)$, there exists a constant $M > 0$ such that we can write down the majorant function $F(\tilde{y})$ as

$$F(\tilde{y}) = M \prod_{k=0}^{d} \left(1 - \frac{y_k - y_{k0}}{r}\right)^{-1}, \qquad \tilde{y} \in \bar{B}(\tilde{y}_0, r)$$

Proof. Since f converges in $B(\tilde{y}_0, R)$, then it absolutely converges in $\bar{B}(\tilde{y}_0, r)$. If we select a $\tilde{y} \in \partial B(\tilde{y}_0, r)$ on the boundary, we have the following convergent series

$$\sum_{j \geq 0} |a_j||\tilde{y} - \tilde{y}_0|^j = \sum_{j \geq 0} |a_j| r^{j_0 + \cdots + j_d}$$

Then there exists $M > 0$ such that

$$|a_j| r^{|j|} \leq M, \qquad |a_j| \leq \frac{M}{r^{|j|}}, \qquad |j| = j_0 + \cdots + j_d$$

Now we can construct a majorant function

$$F(\tilde{y}) = \sum_{j \geq 0} \frac{M}{r^{|j|}} (\tilde{y} - \tilde{y}_0)^j = M \prod_{k=0}^{d} \sum_{j_k \geq 0} \left(\frac{y_k - y_{k0}}{r}\right)^{j_k} = M \prod_{k=0}^{d} \left(1 - \frac{y_k - y_{k0}}{r}\right)^{-1} \qquad \blacksquare$$

Corollary. For $F(\tilde{y})$ defined above, consider the Cauchy problem

$$y_j' = F(z, y), \qquad y_j(z_0) = y_{j0}, \qquad j = 1, 2, \cdots, d$$

There exists $\rho > 0$ such that it has a unique analytic solution on $B(z_0, \rho)$.

Proof. Denote $u(z) = y_1(z) - y_{10}$. Based on the Cauchy problem, we have

$$(y_i - y_j)' = 0, \qquad u(z) = y_i(z) - y_{i0}, \qquad \forall i, j = 1, 2, \cdots, d$$

The ODE for $u(z)$ can be obtained as

$$u'(z) = F(z, y) = M \left(1 - \frac{z - z_0}{r}\right)^{-1} \left(1 - \frac{u}{r}\right)^{-d}, \qquad u(z_0) = 0$$

The solution is

$$u(z) = r - r\left[1 + (d+1)M \ln\left(1 - \frac{z - z_0}{r}\right)\right]^{\frac{1}{d+1}}$$

To guarantee convergence, we can obtain the radius ρ as

$$\left|\frac{z - z_0}{r}\right| < 1, \qquad \left|(d+1)M \ln\left(1 - \frac{z - z_0}{r}\right)\right| < 1, \qquad \rho = r\left[1 - e^{-\frac{1}{(d+1)M}}\right] \qquad \blacksquare$$

Cauchy theorem

Let $\Omega \subseteq \mathbb{C}^{d+1}$ and denote analytic functions $f: \Omega \to \mathbb{C}^d$ with a point $(z_0, y_0) \in \Omega$. There exists $\rho > 0$ such that the Cauchy problem has a unique analytic solution in $B(z_0, \rho)$.

$$y_j' = f_j(z, y), \qquad y_j(z_0) = y_{j0}, \qquad j = 1,2,\cdots,d$$

Proof. Without loss of generality, assume $z_0 = 0$ and $y_0 = 0$. We consider the solution in the form of a power series

$$f_j(\tilde{y}) = f_j(z, y) = \sum_{J \geq 0} a_{jJ} \tilde{y}^J, \qquad y_j(z) = \sum_{m \geq 0} c_{jm} z^m, \qquad j = 1,2,\cdots,d$$

Then we have

$$y_j' = \sum_{k \geq 0} c_{j(k+1)} (k+1) z^k = \sum_{J \geq 0} a_{jJ} z^{J_0} y_1^{J_1} \cdots y_d^{J_d} = f_j(\tilde{y})$$

Substitute $y_j(z)$ into the RHS and compare the coefficients. We can obtain

$$c_{jm} = P_{jm}\big(a_{jJ} \mid |J| \leq m-1\big)$$

The polynomial P_{jm} has positive coefficients. To prove the solution is convergent, consider

$$\hat{y}_j' = F(z, y) = M \prod_{k=0}^{d} \left(1 - \frac{y_k}{r}\right)^{-1}$$

Here M is sufficiently large such that $F(z, y)$ is the majorant function for all $f_1, \cdots, f_d$. We have

$$\hat{y}_j(z) = \sum_{m \geq 0} \hat{c}_{jm} z^m, \qquad \hat{c}_{jm} = P_{jm}\big(A_J \mid |J| \leq m-1\big)$$

A_J is the coefficient for the majorant series $F(\tilde{y})$. Since

$$|c_{jm}| = |P_{jm}(a_{jJ})| \leq P_{jm}(|a_{jJ}|) \leq P_{jm}(A_J) = \hat{c}_{jm}$$

Therefore, the formal series $y_j(z)$ converges. ■

Corollary. For the matrix equation

$$Y'(z) = A(z) Y(z), \qquad Y(z_0) = I$$

If A is analytic near z_0, then there exists a unique analytic solution.

Theorem. Consider $F \in M(\mathbb{O}(\Omega), d)$ with the following equation and its formal solution

$$z y' = F y, \qquad y(z) = \sum_{k \geq 0} c_k (z - z_0)^k, \qquad c_k \in \mathbb{C}^n$$

There exists $\rho > 0$ such that $y(z)$ converges in $B(z_0, \rho)$ and thus is an analytic solution.

Proof. Assume $z_0 = 0$. Consider $F(z)$ can be expanded as

$$F(z) = \sum_{k\geq 0} F_k z^k, \qquad F_k \in M(\mathbb{C}, d) = \mathbb{C}^{d\times d}$$

The equation becomes

$$\sum_{m\geq 0} mc_m z^m = \left(\sum_{k\geq 0} F_k z^k\right)\left(\sum_{l\geq 0} c_l z^l\right) = \sum_{m\geq 0}\left(\sum_{k+l=m} F_k c_l\right) z^m$$

Comparing the coefficients gives

$$mc_m = \sum_{k+l=m} F_k c_l, \qquad (F_0 - mI)c_m = -\sum_{k=1}^{m} F_k c_{m-k}$$

For $m = 0$ we have $F_0 c_0 = 0$. While for $m = 1$, we have

$$c_1 = F_0 c_1 + F_1 c_0, \qquad (F_0 - I)c_1 = -F_1 c_0$$

If F_0 does not have 1 as an eigenvalue, we can obtain the unique solution of c_1. We take $k \in \mathbb{N}$ that is sufficiently large such that for all $\lambda > k$, the matrix $F_0 - \lambda I$ is invertible. Denote

$$f(\lambda) = |(F_0 - \lambda I)^{-1}|_\infty, \qquad \lambda > k$$

Then we have $f \in C(k, +\infty)$ continuous, and when $\lambda \to +\infty$ we have $f(\lambda) \to 0$. This implies that there exists $C > 0$ such that $f(m) \leq C$ for all $m > k$. The coefficients c_m are bounded as

$$|c_m|_\infty = \left|-(F_0 - \lambda I)^{-1}\sum_{k=1}^{m} F_k c_{m-k}\right|_\infty \leq C\sum_{k=1}^{m}|F_k|_\infty |c_{m-k}|_\infty$$

Define $v_m = |c_m|_\infty$ for $m \leq k$, and

$$v_m = C\sum_{j=1}^{m}\left|F_j\right|_\infty v_{m-j}, \qquad m > k$$

This guarantees $|c_m| \leq v_m$. We want to show that $\{v_m\}$ corresponds to a convergent series.

$$u(z) = \sum_{m\geq 0} v_m z^m, \qquad \phi(z) = \sum_{m\geq 1}|F_m|_\infty z^m$$

We can show that (all norms are $|\cdot|_\infty$)

$$u(z) = [1 - C\phi(z)]^{-1}\left[|c_0| + \sum_{l=1}^{k}\left(|c_l| - C\sum_{j=1}^{l}\left|F_j\right|\left|c_{l-j}\right|\right)z^l\right]$$

This is proved by comparing the coefficients, after multiplying $1 - c\phi(z)$ to the LHS.

$$[z^m]: \quad v_m - C\sum_{j=1}^{m}\left|F_j\right|v_{m-j} = |c_m| - C\sum_{j=1}^{m}\left|F_j\right|\left|c_{m-j}\right|, \qquad m \leq k$$

$$[z^m]: \quad v_m - C\sum_{j=1}^{m}\left|F_j\right|v_{m-j} = 0, \qquad m > k$$

The numerator of $u(z)$ is a polynomial which is convergent. As $\phi(0) = 0$, there exists $\delta_1 > 0$ such that when $|z| < \delta_1$, we have $1 - C\phi(z) \neq 0$ and $(1 - C\phi(z))^{-1}$ is analytic on $B(0, \delta_1)$. Therefore, we prove the majorant $u(z)$ is analytic, and thus $y(z)$. ■

- Asymptotic behavior near ordinary and regular singular points (6.2)

$$zy'(z) = F(z)y(z), \qquad F \in M(B(0,1), n)$$

Now consider the matrix equation

$$zY'(z) = A(z)Y(z), \qquad A \in M(\mathbb{O}(\Omega), n)$$

We require $A(0) \neq 0$ which implies that $z = 0$ is a singular point. The domain $\Omega: |z| < \rho$. Our goal is to find a transform $P \in GL(\mathbb{O}(\Omega), n)$ such that $Y = PX$ and

$$zX'(z) = B(z)X(z), \qquad B = P^{-1}AP - zP^{-1}P'$$

We want to choose B to be as simple as possible. The matrix equation to be solved is

$$zP'(z) = A(z)P(z) - P(z)B(z)$$

With the formal power series, the equation becomes

$$\sum_{m\geq 0} mP_m z^m = \left(\sum_{k\geq 0} A_k z^k\right)\left(\sum_{l\geq 0} P_l z^l\right) - \left(\sum_{l\geq 0} P_l z^l\right)\left(\sum_{k\geq 0} B_k z^k\right)$$

Taking the coefficient of z^m, we obtain

$$mP_m = A_0P_m - P_mB_0 + \sum_{k=1}^{m}(A_kP_{m-k} - P_{m-k}B_k)$$

$$(A_0 - mI)P_m - P_mB_0 = \sum_{k=1}^{m}(P_{m-k}B_k - A_kP_{m-k})$$

For $m = 0$, we have $B_0 = P_0^{-1}A_0P_0$. One choice is $P_0 = I$ and $B_0 = A_0$. Another better one is to choose P_0 such that $B_0 = J_0$ is the Jordan normal form of A_0.

Corollary 1. For $A, B \in M(\mathbb{C}, n)$, define the following map

$$\varphi_{AB}\colon\ M(\mathbb{C}, n) \to M(\mathbb{C}, n), \qquad P \mapsto AP - PB$$

Then φ_{AB} is injective if and only if A, B do not share the same eigenvalue.

Proof. When φ_{AB} is injective, assume that λ is the common eigenvalue. Then there exist non-zero $v, w \in \mathbb{C}^n$ such that $Av = \lambda v$ and $B^Tw = \lambda w$. Take $P = vw^T$, and we obtain

$$AP - PB = Avw^T - vw^TB = \lambda vv^T - \lambda vv^T = O$$

This is contradictory to φ_{AB} being injective. On the other hand, when there are no common eigenvalues between A and B, denote $V = \mathbb{C}^n$ and we can write

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}, \qquad V_{\lambda_j} = \mathrm{Ker}\left(B - \lambda_j I\right)^{m_j}$$

We can obtain a basis for V by picking from each root subspace V_λ. Let v be the basis of V_λ. If P satisfies $AP - PB = O$, we have

$$(B - \lambda I)^m v = 0, \qquad P(B - \lambda I)^m v = (A - \lambda I)^m P v = 0$$

Note that λ is not the eigenvalue of A, so $(A - \lambda I)^m$ is invertible and thus $Pv = 0$. Since v can be any vector of the basis, $P = O$ and thus φ_{AB} is injective. ■

Resonant matrix

With this corollary, we need to see if $A_0 - mI$ and B_0 share the same eigenvalue. We call a matrix A as resonant if there are two eigenvalues λ, μ such that $\lambda - \mu \in \mathbb{Z}_{>0}$.

Theorem 2. For $zY' = AY$, if A_0 is non-resonant, then there exists a transformation $Y = PX$ with $P(0) = I$ and $P(z)$ analytic around $z = 0$ such that

$$zX' = A_0 X, \qquad Y(z) = P(z) z^{A_0}$$

Proof. Since A_0 is non-resonant and $P_0 = I$, we know that $A_0 - mI$ and $B_0 = A_0$ do not share the same eigenvalue. We can then choose $B_m = O$ for $m \geq 1$, and there are corresponding P_m

$$P_m = \varphi_{A_0 - mI, B_0}^{-1} \left(-\sum_{k=1}^{m} A_k P_{m-k} \right), \qquad zX' = A_0 X$$

Therefore, we obtain a formal solution $Y(z) = P(z) z^{A_0}$. The equation for $P(z)$ is

$$zP'(z) = A(z)P(z) - P(z)A_0$$

Now take a basis $e_1, \dots, e_{n^2}$ for $M(\mathbb{C}, n)$, and we have

$$P(z) = \sum_{j=1}^{n^2} p_j e_j, \qquad zP'(z) = M(z)P(z), \qquad M(z): \varphi_{A(z), A_0}$$

From the existence of an analytic solution for the matrix equation, $P(z)$ is analytic at $z = 0$. ■

If A_0 is resonant, then $(A_0 - mI)P_m - P_m B_0$ is not an isomorphism, so we cannot ensure the existence of P_m for arbitrarily chosen B_m.

$$(A_0 - mI)P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k})$$

As an example, we can choose

$$\sum_{k=1}^{m-1} (P_{m-k} B_k - A_k P_{m-k}) + B_m - A_m = O, \qquad P_m = O$$

In this case, we can obtain the following solution.

Proposition 3. For $zY' = AY$, we have a resonant A_0. Let M be the largest positive integer such that $M = \lambda - \mu$ for the eigenvalues. Then there exists $Y = PX$ with analytic $P(z)$ such that

$$zX' = \left(A_0 + \sum_{k=1}^{M} B_k z^k\right) X$$

B_k is non-zero only when there are eigenvalues such that $k = \lambda - \mu$.

A better choice is given as follows. For $zY' = AY$, consider $A_0 = P_0 J_0 P_0^{-1}$ with J_0 as the Jordan normal form. Take $Y = P_0 X$, and then we have

$$zX = (P_0^{-1} A P_0) X = (J_0 + A_1 z + \cdots + A_m z^m) X$$

Without loss of generality, assume $A_0 = \Lambda + N_0$ already the Jordan normal form ($P_0 = I$), and its eigenvalues are ordered by decreasing Re λ_α. As N_0 is strictly upper triangular, we have

$$(N_0)_{\alpha\beta} = 0, \qquad \alpha \geq \beta, \qquad (N_0)_{\alpha\beta} \neq 0, \qquad \lambda_\alpha \neq \lambda_\beta$$

When $m = 1$, the matrix equation becomes

$$(A_0 - I)P_1 - P_1 A_0 = B_1 - A_1$$

Using Einstein summation notation, the (α, β) element becomes

$$\Lambda_{\alpha\gamma}(P_1)_{\gamma\beta} + (N_0)_{\alpha\gamma}(P_1)_{\gamma\beta} - (P_1)_{\alpha\beta} - (P_1)_{\alpha\gamma}(\Lambda)_{\gamma\beta} - (P_1)_{\alpha\gamma}(N_0)_{\gamma\beta} = (B_1)_{\alpha\beta} - (A_1)_{\alpha\beta}$$

For the diagonal matrix, we have $\Lambda_{ij} = \lambda_i \delta_{ij}$, which leads to

$$(\lambda_\alpha - \lambda_\beta - 1)(P_1)_{\alpha\beta} + (N_0)_{\alpha\gamma}(P_1)_{\gamma\beta} - (P_1)_{\alpha\gamma}(N_0)_{\gamma\beta} = (B_1)_{\alpha\beta} - (A_1)_{\alpha\beta}$$

For $(n, 1)$ element, we have $(N_0)_{n\gamma} = (N_0)_{\gamma 1} = 0$, which leads to

$$(\lambda_n - \lambda_1 - 1)(P_1)_{n1} = (B_1)_{n1} - (A_1)_{n1}$$

If $\lambda_n - \lambda_1 \neq 1$, we can choose

$$(B_1)_{n1} = 0, \qquad (P_1)_{n1} = -\frac{(A_1)_{n1}}{\lambda_n - \lambda_1 - 1}$$

If $\lambda_n - \lambda_1 = 1$, we can choose $(B_1)_{n1} = (A_1)_{n1}$, and $(P_1)_{n1}$ is arbitrary. We can continue this process for $(n, 2)$ element and so on using previously determined P_1. This implies that we can find B_1 and P_1, and $(B_1)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = 1$. In general, B_m and P_m exist, and $(B_m)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = m$.

Proposition 3'. With this new choice of B_m (now denoted as N_m) and P_m, we have

$$zX' = (\Lambda + N_0 + N_1 z + \cdots + N_m z^m) X$$

$(N_k)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = k$. This implies that non-zero elements are possible only when $i < j$ since we have ordered the eigenvalues, and N_k are strictly upper triangular.

Corollary 4. With this new choice of Λ and N_k, we have

$$z^{\Lambda}N_k = N_k z^k z^{\Lambda}$$

Proof. Note that

$$(\lambda_\alpha - \lambda_\beta - k)(N_k)_{\alpha\beta} = 0, \qquad \Lambda N_k - N_k\Lambda - kN_k = O$$

Therefore, we have

$$z^{\Lambda}N_k = \sum_{l\geq 0}\frac{(\ln z)^l}{l!}\Lambda^l N_k = \sum_{l\geq 0}\frac{(\ln z)^l}{l!}N_k(\Lambda+k)^l = N_k z^{\Lambda+k} \qquad \blacksquare$$

Corollary 5. For the following equation

$$zX' = (\Lambda + N_0 + N_1 z + \cdots + N_m z^m)X$$

Its solution is

$$\xi = z^{\Lambda}z^{N}, \qquad N = N_0 + N_1 + \cdots + N_m$$

Proof. Using the previous Corollary, we can directly calculate

$$z\xi' = (\Lambda z^{\Lambda})z^N + z^{\Lambda}(Nz^N) = \Lambda\xi + (N_0 + N_1 z + \cdots + N_m z^m)\xi \qquad \blacksquare$$

Theorem 6. For matrix equation $zY' = AY$, assume that A_0 has a Jordan normal form $\Lambda + N_0$, with the eigenvalues ordered by Re λ_α. Then there exists $P(z) \in GL(\mathbb{O}(\Omega), n)$ and a strictly upper triangular constant matrix $N \in M(\mathbb{C}, n)$ such that

$$Y(z) = P(z)z^{\Lambda}z^{N}$$

$(N)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$.

To calculate the solution, since N is a nilpotent matrix with $N^{n+1} = O$, we have

$$z^{\Lambda} = \operatorname{diag}(z^{\lambda_\alpha}), \qquad z^N = \sum_{l=0}^{n}\frac{(\ln z)^l}{l!}N^l$$

For any $\theta_0 \in \mathbb{R}$, the solution $Y(z)$ is analytic in $S(\theta_0) = \{z \in \Omega \mid \theta_0 < \arg z < \theta_0 + 2\pi\}$.

Consider $z \to ze^{2\pi i}$, the solution becomes

$$Y(ze^{2\pi i}) = P(z)z^{\Lambda}e^{2\pi i\Lambda}z^{N}e^{2\pi i N}$$

From Corollary 4, with $z = e^{2\pi i}$ we have

$$e^{2\pi i\Lambda}N_k = N_k e^{2\pi i k}e^{2\pi i\Lambda} = N_k e^{2\pi i\Lambda}, \qquad e^{2\pi i\Lambda}N = Ne^{2\pi i\Lambda}$$

This shows that $e^{2\pi i\Lambda}$ commutes with N, and thus $M = e^{2\pi i\Lambda}e^{2\pi iN}$. Based on this property, we call M the **monodromic matrix** of the matrix equation, and (Λ, N) the **monodromic data** that determine the multivalued properties of the solution.

Example: Bessel equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

We define the vector Y as

$$Y = \begin{bmatrix} y \\ xy' \end{bmatrix}, \qquad Y' = \begin{bmatrix} y' \\ xy'' + y' \end{bmatrix} = \begin{bmatrix} y' \\ \dfrac{\alpha^2 - x^2}{x}y \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{x} \\ \dfrac{\alpha^2 - x^2}{x} & 0 \end{bmatrix}\begin{bmatrix} y \\ xy' \end{bmatrix}$$

Then we obtain the corresponding matrix equation

$$xY' = A(x)Y, \qquad A(x) = \begin{bmatrix} 0 & 1 \\ \alpha^2 - x^2 & 0 \end{bmatrix}$$

The coefficients of the power series of $A(x)$ are

$$A_0 = \begin{bmatrix} 0 & 1 \\ \alpha^2 & 0 \end{bmatrix}, \qquad A_1 = O, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

To diagonalize A_0 (which also set $P_0 = I$), consider the following transform

$$\Phi = \begin{bmatrix} xy' + \alpha y \\ xy' - \alpha y \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ \alpha & -1 \end{bmatrix}Y, \qquad x\Phi' = \left(\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} + \frac{x^2}{2\alpha}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)\Phi$$

When A_0 is non-resonant, we have $2\alpha \notin \mathbb{Z}$ and the solution can be obtained from Theorem 2.

When A_0 is resonant with $2\alpha \in \mathbb{Z}$:

If 2α is odd, as $A_1 = O$ we can choose $B_1 = P_1 = O$, and then for all $m \geq 2$ we can similarly set $B_m = O$ and solve for P_m. The solution can still be written as $Y(z) = P(z)z^{\Lambda}$.

If 2α is even, for $m = 2$ the equation is

$$(A_0 - 2I)P_2 - P_2A_0 = B_2 - A_2$$

As an example, consider $\alpha = 1$ and we have

$$\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}P_2 - P_2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = B_2 - \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Explicitly writing out the elements for P_2, we have

$$P_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} -2a & 0 \\ -4c & -2d \end{bmatrix} = B_2 - \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

This constrains $(B_2)_{12}$ and a valid choice is

$$B_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad P_2 = \frac{1}{8}\begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix}$$

For $m \geq 3$ we can still set $B_m = O$ and solve for P_m. This implies that the final matrix N is

$$N = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l = \begin{bmatrix} 1 & \frac{1}{2}\ln z \\ 0 & 1 \end{bmatrix}, \qquad z^\Lambda = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

We know that for $\alpha = 1$ the solutions are $J_1(z)$ and $Y_1(z)$. The term $\ln z$ contributes to $Y_1(z)$.

In general, for a linear ODE of the form

$$x^n y^{(n)} + p_1(x) x^{n-1} y^{(n-1)} + \cdots + p_n(x) y = 0$$

We can choose vector Y as

$$Y = \left[y, xy', x^2 y'', \cdots, x^{n-1} y^{(n-1)}\right]^T$$

For each element y_j, we can obtain the recursive relation

$$y_j = x^{j-1} y^{(j-1)}, \qquad x y_j' = (j-1) y_j + y_{j+1}$$

This leads to the matrix equation

$$xY' = A(x)Y = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ & & & & n-2 & 1 \\ -p_n(x) & \cdots & \cdots & \cdots & -p_2(x) & n-1-p_1(x) \end{bmatrix} Y$$

Global problem

Consider the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and the only meromorphic functions on $\hat{\mathbb{C}}$ are the rational functions, denoted as $\mathbb{K}$. For matrix $A \in M(\mathbb{K}, n)$ and $Y'(z) = A(z)Y(z)$, we want to know when the equation only has regular singular points. For rational functions, we can write the matrix $A(z)$ as

$$A(z) = \sum_{j=1}^{k} \frac{P_j(z)}{\left(z - z_j\right)^{m_j}} + P_0(z), \qquad \deg(p_j) < m_j$$

If z_j are regular, we have $m_j = 1$ and P_j is a constant matrix. To study the behavior at $z = \infty$, consider $w = 1/z$ and

$$\tilde{Y}(w) = Y\left(\frac{1}{w}\right), \qquad \frac{\mathrm{d}\tilde{Y}}{\mathrm{d}w} = -\frac{1}{w^2} Y'\left(\frac{1}{w}\right) = -\frac{1}{w^2} A\left(\frac{1}{w}\right) \tilde{Y}(w)$$

If $z = \infty$ $(w = 0)$ is a regular singularity, we require $w^{-1}A(w^{-1})$ to be analytic at $w = 0$, which is equivalent to $zA(z)$ having a limit as $z \to \infty$, and this requires $P_0(z) = O$. Therefore, if the equation only has regular singularities on $\hat{\mathbb{C}}$, we have $w\tilde{Y}' = \tilde{A}\tilde{Y}$ with

$$A(z) = \sum_{j=1}^{k} \frac{P_j}{z - z_j}, \qquad \tilde{A}(w) = \sum_{j=1}^{k} \frac{P_j}{w z_j - 1}, \qquad \tilde{A}(0) = -\sum_{j=1}^{k} P_j$$

We can then use a linear fractional transformation to obtain

$$Y'(z) = A(z)Y(z), \qquad A(z) = \sum_{j=1}^{N} \frac{A_j}{z - z_j}, \qquad \sum_{j=1}^{N} A_j = 0$$

Now $z = \infty$ is regular. The singularities z_j decompose $\mathbb{C}$ into simply connected polygons U_α, and there is an analytic solution of the equation in each of them. Every side $\overline{z_j z_k}$ corresponds to a monodromic matrix M_{jk} and thus define a map $(A_j) \mapsto (M_{jk})$, which is related to the Riemann-Hilbert problem.

- Asymptotic behavior near irregular singular points (6.3)

$$z^{r+1}Y'(z) = A(z)Y(z), \qquad r \in \mathbb{N}^*, \qquad r \geq 1$$

We call r as the Poincaré rank, and recall the following classification:

$r = -1$: Ordinary point $\qquad r = 0$: Regular singularity $\qquad r \geq 1$: Irregular singularity

First, we can consider the scalar equation with dimension $n = 1$. We have

$$a(z) = \sum_{k\geq 0} a_k z^k, \qquad \frac{y'}{y} = \frac{a(z)}{z^{r+1}} = \sum_{k=0}^{r-1} \frac{a_k}{z^{r+1-k}} + \frac{a_r}{z} + \sum_{k\geq r+1} a_k z^{k-r-1}$$

The solution is

$$\ln y(z) = \sum_{k=0}^{r-1} \frac{a_k}{k-r} \frac{1}{z^{r-k}} + a_r \ln z + \sum_{k\geq r+1} \frac{a_k}{k-r} z^{k-r}, \qquad y(z) = P(z)\, z^{\rho}\, e^{Q(z^{-1})}$$

The exponent is $\rho = a_r$, and the analytic function $P(z)$ and polynomial $Q(w)$ are defined as

$$P(z) = \exp\left(\sum_{k\geq r+1} \frac{a_k}{k-r} z^{k-r}\right), \qquad Q(w) = \sum_{k=0}^{r-1} \frac{a_k}{k-r} w^{r-k}$$

For the matrix case, we still want to find a transformation P such that $Y = PX$ with

$$z^{r+1}X'(z) = B(z)X(z), \qquad B(z) = P^{-1}AP - z^{r+1}P^{-1}P'$$

The matrix $B(z)$ should be as simple as possible. For the equation of $B(z)$, we similarly obtain

$$z^{r+1}P' = AP - PB$$

Written in formal power series, the coefficients for z^m are

$$[z^m]\, z^{r+1}P' = [z^m] \sum_{k\geq 0} kP_k z^{k+r} = (m-r)P_{m-r}, \qquad P_j = O \text{ for } j < 0$$

$$[z^m]\, (AP - PB) = \sum_{k=0}^{m} (A_k P_{m-k} - P_{m-k} B_k)$$

Therefore, we obtain the following set of equations

$$A_0 P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k}) + (m-r) P_{m-r}$$

We want to properly choose B_m to make the equations simple. Consider A_0 is already reduced to its Jordan normal form, which also gives $P_0 = I$ and $B_0 = A_0$. We need to iteratively solve the matrix equation of the form

$$A_0 P_m - P_m A_0 = B_m - A_m + \sum_{k=1}^{m-1} (P_{m-k} B_k - A_k P_{m-k}) = B_m + F_m$$

The LHS is always resonant. For simplicity, we assume that A_0 has n different eigenvalues and is already diagonalized as $A = \lambda_i \delta_{ij}$. For each element (α, β), we have

$$(\lambda_\alpha - \lambda_\beta)(P_m)_{\alpha\beta} = (B_m)_{\alpha\beta} + (F_m)_{\alpha\beta}$$

When $\alpha \neq \beta$ (off-diagonal elements), we can choose

$$(B_m)_{\alpha\beta} = 0, \qquad (P_m)_{\alpha\beta} = \frac{(F_m)_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}$$

When $\alpha = \beta$ (diagonal elements), we can choose

$$(B_m)_{\alpha\beta} = -(F_m)_{\alpha\beta}, \qquad (P_m)_{\alpha\beta} = 0$$

Theorem 1. For the matrix equation $z^{r+1} Y'(z) = A(z) Y(z)$, consider that A_0 has n different eigenvalues. There exist an invertible $P(z)$ and a diagonal $B(z)$ such that $Y = PX$ and

$$z^{r+1} X'(z) = B(z) X(z)$$

Corollary 2. With a diagonal $B(z)$, similar to the scalar case, we can define

$$Q(w) = \sum_{k=0}^{r-1} \frac{B_k}{k-r} w^{r-k}, \qquad \rho = B_r, \qquad F'(z) = \left(\sum_{k \geq r+1} B_k z^{k-r-1} \right) F(z), \qquad F(0) = I$$

Note that ρ is a constant diagonal matrix, $Q(w)$ is a diagonal matrix with each element being a polynomial of degree r. Then the solution can be written as

$$X(z) = F(z)\, z^\rho\, e^{Q(z^{-1})}, \qquad Y(z) = P(z)\, F(z)\, z^\rho\, e^{Q(z^{-1})}$$

The result uses the property that ρ and Q are commutable since they are diagonal.

Theorem 3. For an analytic $A(z)$ with rank $r \geq 1$, consider that A_0 has n different eigenvalues. The formal solution of the matrix equation $z^{r+1} Y'(z) = A(z) Y(z)$ is given as

$$Y(z) = \hat{Y}(z)\, z^\rho\, e^{Q(z^{-1})}, \qquad \hat{Y}(0) = I$$

For arbitrary $\theta_1, \theta_2 \in \mathbb{R}$ with $0 < \theta_1 - \theta_2 < \pi/r$, there exists $R > 0$ such that the equation has an analytic solution in $S(\theta_1, \theta_2) \cap B(0, R)$, where $S(\theta_1, \theta_2)$ denotes the sector

$$S(\theta_1, \theta_2) = \{z \in \mathbb{C} \mid \theta_1 < \arg z < \theta_2\}$$

Also, as $z \to 0$ within the domain $S(\theta_1, \theta_2)$, the asymptotic behavior should be interpreted as

$$Y(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \tilde{Y}(z)$$

For an irregular singularity at $z = \infty$, similarly consider $w = 1/z$ and we have

$$\tilde{Y}(w) = Y\left(\frac{1}{w}\right), \qquad w^{-r+1}\, \tilde{Y}'(w) = \tilde{A}(w) Y(w), \qquad \tilde{A}(w) = -A\left(\frac{1}{w}\right)$$

The formal solution can be written as

$$\tilde{Y}(w) = \hat{Y}(w)\, w^{\rho}\, e^{Q(w)}$$

The solution is analytic within the domain $S(\theta_1, \theta_2) \cap \{w \in \mathbb{C} \mid |w| > R\}$.

Theorem (Sibuya 1962). For $\theta_0 \in \mathbb{R}$, there exists a sufficiently small $\delta > 0$ such that there is a solution $Y(z)$ in $S(\theta_0 - \delta, \theta_0 + \pi/r) \cap B(0, R)$.

Corollary 4. There exists $\delta > 0, R > 0$ such that there is a solution $Y(z)$ satisfying Theorem 3 in the following domain

$$S_l = \left\{z \in \mathbb{C}^* \mid \frac{\pi}{r}(l-1) - \delta < \arg z < \frac{\pi}{r} l\right\} \cap B(0, R), \qquad l = 1, 2, \cdots, 2r$$

Stokes phenomenon

Now consider the intersection

$$S(l, l+1) = \left\{z \in \mathbb{C}^* \mid \frac{\pi}{r} l - \delta < \arg z < \frac{\pi}{r} l\right\} \cap B(0, R)$$

Corollary 4 indicates that there are solutions Y_l and Y_{l+1} in this domain $S(l, l+1)$. Hence, there is a constant matrix C_l, the **Stokes multiplier**, such that $Y_{l+1}(z) = Y_l(z) C_l$.

In S_l and S_{l+1} respectively, we have

$$Y_l(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \hat{Y}(z), \qquad Y_{l+1}(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \hat{Y}(z), \qquad z \to 0$$

Multiply the second equation with the inverse of the first one, and we have

$$e^{Q(z^{-1})} z^{\rho} C_l\, z^{-\rho} e^{-Q(z^{-1})} \sim I, \qquad z \to 0, \qquad z \in S(l, l+1)$$

For each element (α, β), we have

$$(C_l)_{\alpha\beta}\, e^{q_\alpha(z^{-1}) - q_\beta(z^{-1})}\, z^{\rho_\alpha - \rho_\beta} \sim \delta_{\alpha\beta}, \qquad z \to 0, \qquad z \in S(l, l+1)$$

When $\alpha = \beta$ (diagonal), we have $(C_l)_{\alpha\alpha} = 1$. When $\alpha \neq \beta$ (off-diagonal), note that

$$q_\alpha(z^{-1}) - q_\beta(z^{-1}) = \frac{\lambda_\beta - \lambda_\alpha}{r} z^{-r} + o(z^{-r})$$

Consider a ray $\gamma \in S(l, l+1)$. If there exists a ray γ such that as $z \to 0$ along γ, we have

$$\mathrm{Re}\{(\lambda_\beta - \lambda_\alpha) z^{-r}\} > 0, \qquad \text{then } (C_l)_{\alpha\beta} = 0$$

If there does not exist such a ray γ for the exponent, then nothing can be said about $(C_l)_{ij}$. If the eigenvalues λ_n are sorted by lexicographic order (λ_R, λ_I), then C_l must be an upper or lower triangular matrix, dependent on l being odd or even.

We define the **Stokes ray** as those that lead to

$$\mathrm{Re}\{(\lambda_\beta - \lambda_\alpha) z^{-r}\} = 0$$

There are $M = n(n-1)r$ Stokes rays in total. Each ray corresponds to a Stokes factor.

Example: Airy equation

$$y'' = zy$$

The corresponding matrix equation is

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad Y' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ zy \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} Y$$

To analyze the behavior at $z = \infty$, we rewrite it as

$$z^{-1} Y'(z) = \begin{bmatrix} 0 & 1/z \\ 1 & 0 \end{bmatrix} Y, \qquad r = 2$$

- Exercise

Regular singular point

$$y(z) = \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}, \qquad zy'(z) = \begin{bmatrix} -1/2 + z & z \\ z & 1/2 + z \end{bmatrix} y(z) = A(z) y(z)$$

$z = 0$ is a regular singularity. The coefficients of the power series of $A(z)$ are

$$A_0 = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A_k = O, \qquad k \geq 2$$

We already have a diagonal A_0, which implies $P_0 = I$ and $B_0 = A_0$. For $m = 1$ we have

$$(A_0 - I)P_1 - P_1 A_0 = B_1 - A_1$$

Explicitly writing out the elements for P_1, we have

$$P_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} -a & -2b \\ 0 & -d \end{bmatrix} = B_1 - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We can obtain a lower triangular B_1, as well as the corresponding P_1 as

$$P_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

For $m \geq 2$, there is no resonance and we can choose $B_m = O$. As an example, for $m = 2$

$$(A_0 - 2I)P_2 - P_2 A_0 = B_2 + P_1 B_1 - A_1 P_1$$

We can solve for P_2 as

$$P_2 = \begin{bmatrix} 1/4 & 1/2 \\ 0 & 3/4 \end{bmatrix}, \qquad B_2 = O$$

Repeat this process and we can also obtain

$$P_3 = \begin{bmatrix} -1/12 & 5/16 \\ -1/4 & 5/12 \end{bmatrix}, \qquad B_3 = O$$

The monodromic data (Λ, N) are then given as

$$\Lambda = \operatorname{diag}\left(-\frac{1}{2}, \frac{1}{2}\right), \qquad N = B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The transformation $P(z)$ is given as

$$P(z) = \sum_{k \geq 0} P_k z^k = \begin{bmatrix} 1 + z + \frac{1}{4}z^2 - \frac{1}{12}z^3 + \cdots & \frac{1}{2}z + \frac{1}{2}z^2 + \frac{5}{16}z^3 + \cdots \\ -\frac{1}{4}z^3 + \cdots & 1 + z + \frac{3}{4}z^2 + \frac{5}{12}z^3 + \cdots \end{bmatrix}$$

The fundamental solution matrix becomes

$$Y(z) = P(z) z^{\Lambda} z^{N}, \qquad z^{\Lambda} = \operatorname{diag}\left(\frac{1}{\sqrt{z}}, \sqrt{z}\right), \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l = \begin{bmatrix} 1 & 0 \\ \ln z & 1 \end{bmatrix}$$

Bessel equation

We consider the Bessel equation with integer order

$$x^2 y'' + x y' + (x^2 - n^2) y = 0, \qquad n \in \mathbb{N}^*$$

The matrix equation is

$$\phi = \begin{bmatrix} xy' + ny \\ xy' - ny \end{bmatrix}, \qquad x\phi' = \left(\begin{bmatrix} n & 0 \\ 0 & -n \end{bmatrix} + \frac{x^2}{2n} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) \phi$$

We already have a diagonal A_0, which implies $P_0 = I$ and $B_0 = A_0$.

$$A_0 = \begin{bmatrix} n & 0 \\ 0 & -n \end{bmatrix}, \qquad A_1 = O, \qquad A_2 = \frac{1}{2n} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Now A_0 is resonant, and the difference in eigenvalues $2n$ is even. For each order m, we need to solve the following matrix equation

$$(A_0 - mI)P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k})$$

The LHS is injective whenever $m \neq 2n$. We can set $B_m = O$ for $1 \leq m \leq 2n - 1$ and obtain

$$(A_0 - mI)P_m - P_m A_0 = \delta_{m,2n} B_m - A_2 P_{m-2}, \qquad m = 2, 4, \cdots, 2n$$

We also know that B_{2n} is upper triangular, and denote $(B_{2n})_{12} = \kappa_n$. Explicitly writing out the elements of P_m, from this recursive formula we have

$$\begin{bmatrix} -ma_m & (2n-m)b_m \\ -(2n+m)c_m & -md_m \end{bmatrix} = \begin{bmatrix} 0 & \delta_{m,2n}\kappa_n \\ 0 & 0 \end{bmatrix} + \frac{1}{2n}\begin{bmatrix} a_{m-2}-c_{m-2} & b_{m-2}-d_{m-2} \\ a_{m-2}-c_{m-2} & b_{m-2}-d_{m-2} \end{bmatrix}$$

This leads to

$$ma_m = (2n+m)c_m = \frac{1}{2n}(c_{m-2}-a_{m-2}), \qquad m = 2,4,\cdots,2n$$

$$(m-2n)b_m = md_m = \frac{1}{2n}(d_{m-2}-b_{m-2}), \qquad m = 2,4,\cdots,2n-2$$

The special case $m = 2n$ gives

$$b_{2n} = 0, \qquad \kappa_n = 2nd_{2n} = \frac{1}{2n}(d_{2n-2}-b_{2n-2})$$

Note that the monodromic data (Λ, N) are given as

$$\Lambda = A_0 = \begin{bmatrix} n & 0 \\ 0 & -n \end{bmatrix}, \qquad N = B_{2n} = \begin{bmatrix} 0 & \kappa_n \\ 0 & 0 \end{bmatrix}$$

Hence, we only need b_m and d_m. With the initial conditions $b_0 = 0$ and $d_0 = 1$, we have

$$d_m - b_m = \frac{d_{m-2}-b_{m-2}}{m(2n-m)}, \qquad d_{2l} - b_{2l} = \frac{(n-l-1)!}{4^l \cdot l! \cdot (n-1)!}, \qquad l \le n-1$$

Therefore, we obtain

$$\kappa_n = \frac{n}{2^{2n-1}\cdot (n!)^2}, \qquad n \in \mathbb{N}^*$$

The fundamental solution matrix is obtained as

$$\Phi(x) = P(x)x^\Lambda x^N, \qquad x^\Lambda = \begin{bmatrix} x^n & 0 \\ 0 & x^{-n} \end{bmatrix}, \qquad x^N = \sum_{l=0}^{n} \frac{(\ln x)^l}{l!} N^l = \begin{bmatrix} 1 & \kappa_n \ln x \\ 0 & 1 \end{bmatrix}$$

The leading order term is

$$\Phi(x) = \begin{bmatrix} x^n & \kappa_n x^n \ln x \\ 0 & x^{-n} \end{bmatrix}\left(1 + O(x^2)\right), \qquad z \to 0$$

Note that each column of $\Phi(x)$ is a linearly independent solution of $\phi(x)$. We want to see how $\Phi(x)$ is related to the Bessel functions, which satisfy the following recurrence relation

$$\begin{bmatrix} xZ_n'(x) + nZ_n(x) \\ xZ_n'(x) - nZ_n(x) \end{bmatrix} = \begin{bmatrix} xZ_{n-1}(x) \\ -xZ_{n+1}(x) \end{bmatrix}$$

Here $Z_n(x)$ can be either $J_n(x)$ or $Y_n(x)$. Focusing on the leading order term, we have

$$J_n(x) = \frac{x^n}{2^n \cdot n!}\left(1 + O(x^2)\right)$$

$$\pi Y_n(x) = \left[-\frac{2^n(n-1)!}{x^n} + \frac{x^n \ln\frac{x}{2}}{2^{n-1}\cdot n!} - \frac{\psi(1)+\psi(n+1)}{2^n \cdot n!}x^n\right]\left(1 + O(x^2)\right)$$

For $J_n(x)$, it is obvious to see

$$\begin{bmatrix} xJ_{n-1}(x) \\ -xJ_{n+1}(x) \end{bmatrix} = \frac{1}{2^{n-1}(n-1)!}\begin{bmatrix} x^n \\ 0 \end{bmatrix} + O(x^{n+2})$$

For $Y_n(x)$, we can neglect the third term since it is related to x^n, which is the other solution.

For the other two terms x^{-n} and $x^n \ln x$, we have

$$\begin{bmatrix} xY_{n-1}(x) \\ -xY_{n+1}(x) \end{bmatrix} \sim \begin{bmatrix} \dfrac{x^n \ln x}{2^{n-2}(n-1)!} \\ 2^{n+1}n!\, x^{-n} \end{bmatrix} = 2^{n+1}n! \begin{bmatrix} \kappa_n x^n \ln x \\ x^{-n} \end{bmatrix}$$

Hence we check the factor κ_n and show how $\Phi(x)$ is related to $J_n(x)$ and $Y_n(x)$.

Linear fractional transformation

Consider $Y'(z) = A(z)Y(z)$ defined on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with rational function $A(z)$. The linear fractional transformation is given as

$$z = \frac{aw+b}{cw+d}, \qquad \frac{\mathrm{d}z}{\mathrm{d}w} = \frac{ad-bc}{(cw+d)^2}$$

We thus have

$$\tilde{Y}(w) = Y\big(z(w)\big), \qquad \frac{\mathrm{d}\tilde{Y}}{\mathrm{d}w} = \frac{ad-bc}{(cw+d)^2} Y'\big(z(w)\big) = \frac{ad-bc}{(cw+d)^2} A\big(z(w)\big)\tilde{Y}(w)$$

The transformed equation becomes

$$\tilde{Y}'(w) = \tilde{A}(w)\tilde{Y}(w), \qquad \tilde{A}(w) = \frac{ad-bc}{(cw+d)^2} A\left(\frac{aw+b}{cw+d}\right)$$

The new coefficient $\tilde{A}(w)$ is still a rational function.

Asymptotic Analysis of DEs (2): Linear ODE with Parameters

For $x \in I \subseteq \mathbb{R}$ and $y \in \Omega \subseteq \mathbb{R}^n$, consider the following ODE with respect to a small parameter $\varepsilon \in B^*(0, \delta)$ given as

$$F(x, y, y', \varepsilon) = 0, \qquad y(x_0) = y_0$$

We want to study the asymptotic behavior of its solution $y(x; \varepsilon)$ as $\varepsilon \to 0^{\pm}$.

- Formal power series expansion (7.1)

Assume that the solution can be written as

$$y(x; \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$$

The ODE now becomes

$$F(x, y_0 + \varepsilon y_1 + \cdots, y_0' + \varepsilon y_1' + \cdots; \varepsilon) = 0$$

The Taylor expansion with respect to ε around $p_0 = (x, y_0, y_0'; 0)$ is

$$F = F(p_0) + \varepsilon\left[\frac{\partial F}{\partial y}(p_0) y_1 + \frac{\partial F}{\partial y'}(p_0)\, y_1' + \frac{\partial F}{\partial \varepsilon}(p_0)\right]$$

$$+\varepsilon^2\left[\frac{\partial F}{\partial y} y_2 + \frac{\partial F}{\partial y'} y_2' + \frac{1}{2} y_1^T \frac{\partial^2 F}{\partial y \partial y} y_1 + \frac{1}{2} y_1'^T \frac{\partial^2 F}{\partial y' \partial y'} y_1' + \frac{1}{2}\frac{\partial^2 F}{\partial \varepsilon^2}\right] = 0$$

For each order of ε, we have

$$\varepsilon^0:\ F(x, y_0, y_0'; 0) = 0, \qquad y_0 = y_0(x)$$

$$\varepsilon^1:\ y_1' = A(x) y_1 + B_1(x), \qquad A = -\left(\frac{\partial F}{\partial y'}\right)^{-1} \frac{\partial F}{\partial y}, \qquad B_1 = -\left(\frac{\partial F}{\partial y'}\right)^{-1} \frac{\partial F}{\partial \varepsilon}$$

Note that for $[\varepsilon^1]$, the derivatives are evaluated at $(x, y_0(x), y_0'(x), 0)$. For $[\varepsilon^2]$ we have

$$\varepsilon^2:\ y_2' = A(x) y_2 + B_2(x), \qquad B_2(x) = -\left(\frac{\partial F}{\partial y'}\right)^{-1} (\cdots)$$

As long as the fundamental matrix of $y' = A(x) y$ is known, we can recursively solve $y_n(x)$.

There are several issues arising

- F may not be defined at $\varepsilon = 0$.
- $F(x, y_0, y_0', 0)$ may not have a solution (e.g., boundary layer equation).
- The Jacobian $\partial F / \partial y'$ is not invertible at $p_0 = (x, y_0, y_0'; 0)$.
- The properties of the formal power series are bad.

Now simply consider a function $y(x; \varepsilon)$ with its formal power series

$$y(x; \varepsilon) = \sum_{n \geq 0} y_n(x) \varepsilon^n, \qquad \varepsilon \to 0$$

The equivalent statement is that for $\forall N \in \mathbb{N}$ we have

$$\lim_{\varepsilon\to 0}\frac{y(x;\varepsilon)-\sum_{n=0}^{N} y_n(x)\varepsilon^n}{y_N(x)\varepsilon^N}=0$$

If the function series has pointwise but not uniform convergence, then the remainder depends on x and is unbounded at some points. The partial sum is thus not practical to use.

Example: Duffing equation

$$y''+y+\varepsilon y^3=0,\qquad y(0)=1,\qquad y'(0)=0$$

Multiplying y' gives

$$\left(\frac{1}{2}y'^2+\frac{1}{2}y^2+\frac{\varepsilon}{4}y^4\right)'=0,\qquad (y')^2+y^2+\frac{\varepsilon}{2}y^4=1+\frac{\varepsilon}{2}$$

The constant is determined from the initial conditions. This leads to an elliptical integral

$$x=\pm\int_1^y\frac{\mathrm{d}y}{\sqrt{\left(1+\frac{\varepsilon}{2}\right)-y^2-\frac{\varepsilon}{2}y^4}}$$

We notice that $y_{\min}=-1$ and $y_{\max}=1$. The period of the oscillator is

$$T=2\int_{y_{\min}}^{y_{\max}}\frac{\mathrm{d}y}{\sqrt{\left(1+\frac{\varepsilon}{2}\right)-y^2-\frac{\varepsilon}{2}y^4}}$$

If we directly expand it into a formal power series, we have

$$(y_0''+\varepsilon y_1''+\varepsilon^2 y_2''+\cdots)+(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots)+\varepsilon(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots)^3=0$$

The initial conditions are

$$y_0(0)=1,\qquad y_0'(0)=0,\qquad y_k(0)=0,\qquad y_k'(0)=0,\qquad k\geq 1$$

For each order of ε, we have

$$\varepsilon^0:\ y_0''+y_0=0,\qquad y_0=\cos x$$

$$\varepsilon^1:\ y_1''+y_1+y_0^3=0,\qquad y_1=\frac{1}{32}(\cos 3x-\cos x)-\frac{3}{8}x\sin x$$

The $x\sin x$ term gives an increasing amplitude with x. We can similarly obtain

$$y_2=-\frac{9}{128}x^2\cos x+\frac{3}{32}x\sin x-\frac{9}{256}x\sin 3x+\cdots$$

The **secular terms** such as $x^n\cos x$ make the partial sum useless for computation. The reason for this behavior is the resonance with the forcing term involving y_0 to y_{n-1}. Now we consider a simpler version of the Duffing equation

$$y''+y+\varepsilon y=0,\qquad y(x;\varepsilon)=\cos\left(\sqrt{1+\varepsilon}\,x\right)$$

The period deviates slightly from 2π, and the Taylor expansion will lead to secular terms. This shows the limitation of the method of direct expansion.

- Poincaré-Lindstedt, Poincaré-Lighthill-Kuo (PLK), Strained coordinate method (9.3)

Consider the following example (Tsien, 1956)

$$(x+\varepsilon u)u'+u=0, \qquad u(1)=1$$

First we try using the formal power series

$$u=u_0+\varepsilon u_1+\varepsilon^2 u_2+\cdots$$

For each order of ε, we have

$$\varepsilon^0:\ xu_0'+u_0=0, \qquad u_0(1)=1, \qquad u_0=\frac{1}{x}$$

$$\varepsilon^1:\ xu_1'+u_0u_0'+u_1=0, \qquad u_1(1)=0, \qquad u_1=\frac{1}{2x}\left(1-\frac{1}{x^2}\right)$$

We can similarly obtain

$$\varepsilon^2:\ xu_2'+u_2+u_0u_1'+u_1u_0'=0, \qquad u_2=-\frac{1}{2x^3}\left(1-\frac{1}{x^2}\right)$$

The solution is ordinary around $x=1$, but is singular at $x=0$. In other words, the solution is uniformly convergent in $[a,+\infty)$ for any $a>0$, but not in $(0,+\infty)$.

Strained coordinate (9.3.3)

We introduce the strained coordinate $x=x(\xi)$ with the formal power series

$$u(x;\varepsilon)=u_0(\xi)+\varepsilon u_1(\xi)+\varepsilon^2 u_2(\xi)+\cdots$$

$$x(x;\varepsilon)=\xi+\varepsilon x_1(\xi)+\varepsilon^2 x_2(\xi)+\cdots$$

Now we denote u' and x' as the derivatives with respect to ξ. The operator becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}=\frac{1}{x'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}$$

Then the ODE becomes

$$(x+\varepsilon u)u'+x'u=0$$

For each order of ε, we have

$$\varepsilon^0:\ \xi u_0'+u_0=0, \qquad u_0=\frac{1}{\xi}$$

$$\varepsilon^1:\ \xi u_1'+u_1=-x_1u_0'-x_1'u_0-u_0u_0', \qquad x_1(1)=u_1(1)=0$$

Here both x_1 and u_1 are unknowns. We require that the singularity of u_1 at $\xi=0$ is not higher than the singularity of u_0. We want to find x_1 such that the RHS is ordinary at $\xi=0$. A simple choice is to let the RHS be zero, which gives

$$(\xi u_1)'=-\left(x_1u_0+\frac{1}{2}u_0^2\right)'=0, \qquad x_1=\frac{1}{2}\left(\xi-\frac{1}{\xi}\right), \qquad u_1=0$$

We can similarly obtain

$$\varepsilon^2:\ (x_2+u_1)u_0'+(x_1+u_0)u_1'+\xi u_2'+u_2+u_0x_2'+u_1x_1'=0$$

$$\xi u_2'+u_2=\frac{x_2}{\xi^2}-\frac{x_2'}{\xi}, \qquad x_2(1)=u_2(1)=0$$

The choice $x_2 = u_2 = 0$ is valid. For $n \geq 2$, the equation is homogeneous with respect to x_n, and we can always choose $x_n = u_n = 0$. Hence, we obtain an exact solution

$$u(\xi;\varepsilon) = \frac{1}{\xi}, \qquad x(\xi;\varepsilon) = \xi + \frac{\varepsilon}{2}\left(\xi - \frac{1}{\xi}\right)$$

Writing as $u = u(x;\varepsilon)$, we have

$$u = -\frac{x}{\varepsilon} + \sqrt{\left(\frac{x}{\varepsilon}\right)^2 + \frac{2}{\varepsilon} + 1}$$

Example: Duffing equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

Now we consider the solution

$$y(x;\varepsilon) = y_0(\xi) + \varepsilon y_1(\xi) + \varepsilon^2 y_2(\xi) + \cdots$$
$$x(x;\varepsilon) = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \cdots$$

The second-order derivative operator becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{x'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{y'(\xi)}{x'(\xi)}\right) = \frac{y''x' - y'x''}{(x')^3}$$

The equation then becomes

$$y''x' - y'x'' + (x')^3(y + \varepsilon y^3) = 0$$

The initial conditions are

$$y_0(0) = 1, \qquad y_0'(0) = 0, \qquad y_k(0) = y_k'(0) = 0, \qquad x_k(0) = 0, \qquad k \geq 1$$

There is no constraint on $x_k'(0)$ and we can set $x_k'(0) = 0$. For each order of ε, we have

$$\varepsilon^0\text{:}\ y_0'' + y_0 = 0, \qquad y_0 = \cos\xi$$
$$\varepsilon^1\text{:}\ y_1'' + y_1 = y_0'x_1'' + 2y_0''x_1' - y_0^3$$

Using the solution y_0, we have

$$y_1'' + y_1 = -\sin\xi\, x_1'' - 2\cos\xi\, x_1' - \frac{3}{4}\cos\xi - \frac{1}{4}\cos 3\xi$$

The forcing term $\cos\xi$ leads to resonance, and we want to suppress the secular term by setting

$$\sin\xi\, x_1'' + 2\cos\xi\, x_1' + \frac{3}{4}\cos\xi = 0, \qquad x_1 = -\frac{3}{8}\xi$$

Then the equation for y_1 becomes

$$y_1'' + y_1 = -\frac{1}{4}\cos 3\xi\,, \qquad y_1 = \frac{1}{32}(\cos 3\xi - \cos\xi)$$

We can further obtain (e.g., Fourier series expansion)

$$\varepsilon^2\text{:}\ y_2'' + y_2 = y_1'x_1'' + y_0'x_2'' + 2y_0''x_2' + 2y_1''x_1' - 3y_0''(x_1')^2 - 3y_0'x_0'x_1'' - 3y_0^2 y_1$$
$$= -\sin\xi\, x_2'' - 2\cos\xi\, x_2' + \frac{57}{128}\cos\xi + \frac{1}{16}\cos 3\xi - \frac{3}{128}\cos 5\xi$$

To suppress the secular term, we need to set

$$-\sin\xi\, x_2'' - 2\cos\xi\, x_2' + \frac{57}{128}\cos\xi = 0, \qquad x_2 = \frac{57}{256}\xi$$

With this choice of x_2, we can solve for y_2. Eventually, the solution is

$$x = \xi\left(1 - \frac{3}{8}\varepsilon + \frac{57}{256}\varepsilon^2 - \cdots\right) = \xi\left(1 + \sum_{k\geq 1}\omega_k\varepsilon^k\right)$$

This is the typical form of the strained coordinate for weakly nonlinear oscillations.

➢ Method of multiple scales (9.3.4)

$$y'' + 2\varepsilon y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

The exact solution is obtained from the characteristic equation

$$\lambda^2 + 2\varepsilon\lambda + 1 = 0, \qquad \lambda_{1,2} = -\varepsilon \pm i\sqrt{1-\varepsilon^2}$$

$$y(x) = e^{-\varepsilon x}\cos\left(\sqrt{1-\varepsilon^2}\, x + \theta_0\right), \qquad \theta_0 = -\arctan\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$$

First we try using the strained coordinate method

$$y''x' - y'x'' + 2\varepsilon(x')^2y' + (x')^3y = 0$$

For each order of ε, we have

$$\varepsilon^0\colon\ y_0'' + y_0 = 0, \qquad y_0 = \cos\xi$$

$$\varepsilon^1\colon\ y_1'' + y_1 = -\sin\xi\, x_1'' - 2\cos\xi\, x_1' - 2\sin\xi$$

We still want to suppress the secular term, but now $x_1(\xi)$ becomes singular at $\xi = \pi$

$$x_1 = 1 - \xi\cot\xi$$

The issue is due to the lack of amplitude information ($e^{-\varepsilon x}$) in the strained coordinate method.

In this damped oscillation, there are two (time) scales for the fast oscillation and slow damping, respectively. We introduce a number of scales

$$T_0 = x, \qquad T_1 = \varepsilon x, \qquad T_k = \varepsilon^k x$$

Consider the solution of the form

$$y(x;\varepsilon) = Y(T_0, T_1, \cdots, T_k; \varepsilon) = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$$

The goal is to convert the original ODE into PDEs with more degrees of freedom introduced. With the notations D_x and ∂_k, the derivative operator becomes

$$D_x y = \frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{k\geq 0}\frac{\partial Y}{\partial T_k}\frac{\partial T_k}{\partial x} = \sum_{k\geq 0}\varepsilon^k\frac{\partial Y}{\partial T_k} = \sum_{k\geq 0}\varepsilon^k\partial_k Y$$

The damped oscillation equation then becomes

$$[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2) + 2\varepsilon(\partial_0 + \varepsilon\partial_1) + 1](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) = o(\varepsilon^2)$$

The initial conditions are analyzed as

$$y(0)=Y(T_0(0),T_1(0),\cdots;\varepsilon)=Y(\mathbf{0};\varepsilon)=Y_0(\mathbf{0})+\varepsilon Y_1(\mathbf{0})+\cdots=1$$

$$y'(0)=\left(\sum_{k\geq 0}\varepsilon^k\partial_k\right)\left(\sum_{l\geq 0}\varepsilon^l Y_l(\mathbf{0})\right)=0$$

This leads to the initial conditions for $Y_k(\mathbf{0})$ given as

$$Y_0(\mathbf{0})=1,\qquad Y_k(\mathbf{0})=0,\qquad k\geq 1$$

For the derivatives, the first several orders give the initial conditions

$$\partial_0 Y_0(\mathbf{0})=0,\qquad \partial_1 Y_0(\mathbf{0})+\partial_0 Y_1(\mathbf{0})=0,\qquad \partial_0 Y_2(\mathbf{0})+\partial_1 Y_1(\mathbf{0})+\partial_2 Y_0(\mathbf{0})=0$$

For ε^0 term, we have

$$\varepsilon^0:\ \partial_0^2 Y_0+Y_0=0,\qquad Y_0=A_0(T_1,\cdots)\cos T_0+B_0(T_1,\cdots)\sin T_0$$

For ε^1 term, we have

$$\varepsilon^1:\ \partial_0^2 Y_1+Y_1=-2\partial_1\partial_0 Y_0-2\partial_0 Y_0=2(\partial_1 A_0+A_0)\sin T_0-2(\partial_1 B_0+B_0)\cos T_0$$

To suppress the secular terms, we set the coefficients of the resonant forcing as zero

$$\partial_1 A_0+A_0=0,\qquad \partial_1 B_0+B_0=0$$

We can update the general solutions for A_0 and B_0 as

$$A_0(T_1,\cdots)=e^{-T_1}A_0(T_2,\cdots),\qquad B_0(T_1,\cdots)=e^{-T_1}B_0(T_2,\cdots)$$

The equation of Y_1 then gives

$$\partial_0^2 Y_1+Y_1=0,\qquad Y_1=A_1(T_1,\cdots)\cos T_0+B_1(T_1,\cdots)\sin T_0$$

For ε^2 terms, we can further obtain

$$\partial_0^2 Y_2+Y_2+2(\partial_0\partial_1+\partial_0)Y_1+(2\partial_0\partial_2+\partial_1^2+2\partial_1)Y_0=0$$

Note that from previous results, we already have

$$Y_0=e^{-T_1}A_0(T_2,\cdots)\cos T_0+e^{-T_1}B_0(T_2,\cdots)\sin T_0$$

$$\partial_2\partial_0 Y_0=-e^{-T_1}(\partial_2 A_0)\sin T_0+e^{-T_1}(\partial_2 B_0)\cos T_0\,,\qquad (\partial_1^2+2\partial_1)Y_0=-Y_0$$

$$(\partial_0\partial_1+\partial_0)Y_1=-(\partial_1 A_1+A_1)\sin T_0+(\partial_1 B_1+B_1)\cos T_0$$

The equation for Y_2 is thus obtained as

$$\partial_0^2 Y_2+Y_2=(2\partial_2 A_0+B_0)e^{-T_1}\sin T_0+(-2\partial_2 B_0+A_0)e^{-T_1}\cos T_0$$
$$-(\partial_1 A_1+A_1)\sin T_0+(\partial_1 B_1+B_1)\cos T_0$$

To remove these secular terms, we require

$$\partial_2 A_0=-\frac{1}{2}B_0,\qquad \partial_2 B_0=\frac{1}{2}A_0,\qquad \partial_2^2 A_0+\frac{1}{4}A_0=0$$

$$\partial_1 A_1+A_1=0,\qquad \partial_1 B_1+B_1=0$$

We can update the general solutions of the coefficients as

$$A_0(T_1,\cdots)=A_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)+B_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)$$

$$B_0(T_1,\cdots)=A_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)-B_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)$$

$$A_1(T_1,\cdots) = e^{-T_1}A_1(T_2,\cdots), \qquad B_1(T_1,\cdots) = e^{-T_1}B_1(T_2,\cdots)$$

The equation of Y_2 then gives

$$\partial_0^2 Y_2 + Y_2 = 0, \qquad Y_2 = A_2(T_1,\cdots)\cos T_0 + B_2(T_1,\cdots)\sin T_0$$

As a summary, now we obtain

$$Y_0 = \left[A_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right) + B_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)\right]\cos T_0$$
$$+\left[A_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right) - B_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)\right]\sin T_0$$
$$Y_1 = e^{-T_1}A_1(T_2,\cdots)\cos T_0 + e^{-T_1}B_1(T_2,\cdots)\sin T_0$$

At the initial $x = 0$, we have

$$Y_0(\mathbf{0}) = A_0(T_3,\cdots) = 1, \qquad \partial_0 Y_0(\mathbf{0}) = -B_0(T_3,\cdots) = 0$$
$$Y_1(\mathbf{0}) = A_1(T_2,\cdots) = 0, \qquad \partial_1 Y_0(\mathbf{0}) + \partial_0 Y_1(\mathbf{0}) = -A_0(T_3,\cdots) + B_1(T_2,\cdots) = 0$$

With these coefficients, we have

$$Y_0 = e^{-T_1}\cos\left(T_0 - \frac{1}{2}T_2\right), \qquad Y_1 = e^{-T_1}\sin T_0$$

The summary of the current solution is

$$y(x;\varepsilon) = Y_0 + \varepsilon Y_1 + \cdots = e^{-\varepsilon x}\left[\cos\left(x - \frac{1}{2}\varepsilon^2 x + \cdots\right) + \varepsilon\sin(x+\cdots)\right] + \cdots$$

Example 1: Van der Pol oscillator (p397)

$$y'' + \varepsilon(y^2 - 1)y' + y = 0$$

We want to obtain a general solution. The equation becomes

$$[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2) + \cdots](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots)$$
$$+ \varepsilon(Y_0^2 + 2\varepsilon Y_0 Y_1 - 1)(\partial_0 + \varepsilon\partial_1)(Y_0 + \varepsilon Y_1) + (Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots) = 0$$

For ε^0 term, we still have

$$\varepsilon^0\text{:}\ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_1(T_1,T_2,\cdots)\cos\big(T_0 + B_1(T_1,T_2,\cdots)\big)$$

For ε^1 term, denote $\theta = T_0 + B_1$ and we have

$$\varepsilon^1\text{:}\ \partial_0^2 Y_1 + Y_1 = -2\partial_0\partial_1 Y_0 - (Y_0^2 - 1)\partial_0 Y_0$$
$$= 2(\partial_1 A_1)\sin\theta + 2A_1\cos\theta\,(\partial_1 B_1) + (A_1^2\cos^2\theta - 1)A_1\sin\theta$$
$$= \left(2\partial_1 A_1 - A_1 + \frac{1}{4}A_1^3\right)\sin\theta + 2A_1(\partial_1 B_1)\cos\theta + \frac{1}{4}A_1^3\sin 3\theta$$

To suppress the secular terms, we require

$$2\partial_1 A_1 - A_1 + \frac{1}{4}A_1^3 = 0, \qquad \partial_1 B_1 = 0$$

We can then solve for A_1 and B_1 as

$$\frac{1}{A_1^2} = \frac{1}{4}(C_1 e^{-T_1} + 1), \qquad A_1 = \frac{2}{\sqrt{1 + C_1(T_2,\cdots)e^{-T_1}}}, \qquad B_1 = B_2(T_2,\cdots)$$

Now the equation for Y_1 can be solved as

$$\partial_0^2 Y_1 + Y_1 = \frac{1}{4} A_1^3 \sin 3\theta\,, \qquad Y_1 = -\frac{A_1^3}{32}\sin(3\theta) + C_2 \cos\theta$$

The summary of the current solution is

$$y(x;\varepsilon) = \frac{2}{\sqrt{1 + C_1(\varepsilon^2 x, \cdots) e^{-\varepsilon x}}} \cos\big(x + B_2(\varepsilon^2 x, \cdots)\big) + \varepsilon Y_1 + o(\varepsilon)$$

Example 2: Mathieu equation (9.2)

$$y'' + (\delta(\varepsilon) + \varepsilon \cos x) y = 0$$

We want to properly choose $\delta(\varepsilon)$ such that the solution still has a period of 2π. Consider

$$\delta(\varepsilon) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2 \delta_2 + \cdots$$

Directly expanding $y(x;\varepsilon)$ into the formal power series, we have

$$(y_0'' + \varepsilon y_1'' + \cdots) + (\delta_0 + \varepsilon\delta_1 + \cdots + \varepsilon\cos x)(y_0 + \varepsilon y_1 + \cdots) = 0$$

For ε^0 term, to keep the 2π-periodicity we have

$$\varepsilon^0\!: \; y_0'' + \delta_0 y_0 = 0, \qquad y_0 = A_0 \cos\big(\sqrt{\delta_0}x\big) + B_0 \sin\big(\sqrt{\delta_0}x\big), \qquad \delta_0 = n^2, \quad n \in \mathbb{N}^*$$

Take $\delta_0 = 1$, and for ε^1 term we have

$$\varepsilon^1\!: \; y_1'' + \delta_0 y_1 = -\delta_1 y_0 - y_0 \cos x$$

$$y_1'' + y_1 = -(\delta_1 + \cos x)(A_0 \cos x + B_0 \sin x)$$

To suppress the secular terms, we require $\delta_1 = 0$ and then y_1 is solved as

$$y_1 = -\frac{A_0}{2} + \frac{A_0}{6}\cos 2x + \frac{B_0}{6}\sin 2x + A_1 \cos x + B_1 \sin x$$

For ε^2 term, we have

$$\begin{aligned}\varepsilon^2\!: \; y_2'' + y_2 &= -\delta_2 (A_0 \cos x + B_0 \sin x) \\ &\quad - \cos x \left(-\frac{A_0}{2} + \frac{A_0}{6}\cos 2x + \frac{B_0}{6}\sin 2x + A_1\cos x + B_1 \sin x\right)\end{aligned}$$

To suppress the secular terms, we have

$$A_0\left(-\delta_2 + \frac{5}{12}\right) = 0, \qquad -B_0\left(\delta_2 + \frac{1}{12}\right) = 0$$

Therefore, since the initial conditions determine A_0 and B_0, not all conditions will lead to the same period of 2π. When A_0 or B_0 is zero, it is possible to keep the same period.

Now we study the Mathieu equation using the method of multiple scales. Directly set $\delta_0 = 1$ and $\delta_1 = 0$, and we have

$$\begin{aligned}&[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2)](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) \\ &\qquad + (1 + \varepsilon\cos T_0 + \varepsilon^2\delta_2)(Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) = o(\varepsilon^2)\end{aligned}$$

For ε^0 and ε^1 terms, we have

$$\varepsilon^0:\ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_0(T_1,\cdots)\cos T_0 + B_0(T_1,\cdots)\sin T_0$$

$$\varepsilon^1:\ \partial_0^2 Y_1 + Y_1 = -2(-\partial_1 A_0 \sin T_0 + \partial_1 B_0 \cos T_0) - A_0 \frac{1+\cos 2T_0}{2} - \frac{B_0}{2}\sin 2T_0$$

To suppress the secular terms, we require

$$\partial_1 A_0 = 0, \qquad \partial_1 B_0 = 0, \qquad A_0 = A_0(T_2,\cdots), \qquad B_0 = B_0(T_2,\cdots)$$

The general solution to Y_1 is the same as previous

$$Y_1 = -\frac{A_0}{2} + \frac{A_0}{6}\cos 2T_0 + \frac{B_0}{6}\sin 2T_0 + A_1(T_1,\cdots)\cos T_0 + B_1(T_1,\cdots)\sin T_0$$

For ε^2 term, we have

$$\varepsilon^2:\ \partial_0^2 Y_2 + Y_2 = -2(-\partial_1 A_1 \sin T_0 + \partial_1 B_1 \cos T_0) - 2(-\partial_2 A_0 \sin T_0 + \partial_2 B_0 \cos T_0)$$
$$+\frac{1}{2}A_0\cos T_0 - \frac{1}{12}A_0 \cos T_0 - \frac{B_0}{12}\sin T_0 - \delta_2 A_0 \cos T_0 - \delta_2 B_0 \sin T_0 + \cdots$$

The non-resonant forcing terms are neglected. To suppress the secular terms, we require

$$2\partial_1 A_1 + 2\partial_2 A_0 = \left(\frac{1}{12} + \delta_2\right)B_0, \qquad -2\partial_1 B_1 - 2\partial_2 B_0 = \left(-\frac{5}{12} + \delta_2\right)A_0$$

Note that from ε^1 term, we have A_0 and B_0 depending on T_2 and further. Consider a simpler case with $A_1 = B_1 = 0$, which correspond to specific initial conditions. This gives

$$\partial_2 A_0 = \frac{1}{2}\left(\frac{1}{12} + \delta_2\right)B_0, \qquad \partial_2 B_0 = \frac{1}{2}\left(\frac{5}{12} - \delta_2\right)A_0$$

This leads to a second-order equation for A_0 as

$$\partial_2^2 A_0 + K_2 A_0 = 0, \qquad K_2 = \frac{1}{4}\left(\delta_2 + \frac{1}{12}\right)\left(\delta_2 - \frac{5}{12}\right)$$

Depending on the sign of K_2, we have

$$A_0 = C_1 \cos\sqrt{K_2}T_2 + C_2 \sin\sqrt{K_2}T_2\,, \qquad \delta_2 < -\frac{1}{12}\ \text{ or }\ \delta_2 > \frac{5}{12}$$
$$A_0 = C_1 e^{\sqrt{-K_2}T_2} + C_2 e^{-\sqrt{-K_2}T_2}, \qquad -\frac{1}{12} < \delta_2 < \frac{5}{12}$$

The summary of the current solution is

$$y(x;\varepsilon) = A_0\cos T_0 + B_0 \sin T_0 + \varepsilon Y_1 + \cdots$$

For the exponential case, the finite energy of the system implies $C_1 = 0$, while the exponential decay cannot be observed. This corresponds to the band gap.

➢ Exercise

Method of multiple scales 1

$$u''(t) + \omega^2 u = \varepsilon u^3, \qquad u(0) = a, \qquad u'(0) = 0$$

We introduce a new timescale $\bar{t}$ and the solution of the form

$$\bar{t} = t + \varepsilon f_1(t) + \cdots\,, \qquad u = \sum_{n=0}^{\infty}\varepsilon^n u_n(\bar{t})\,, \qquad \bar{t}(0) = 0$$

The derivative operator becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = [1 + \varepsilon f_1'(t) + \cdots]\frac{\mathrm{d}u}{\mathrm{d}\bar{t}}, \qquad \frac{\mathrm{d}^2u}{\mathrm{d}t^2} = [1 + \varepsilon f_1'(t) + \cdots]^2\frac{\mathrm{d}^2u}{\mathrm{d}\bar{t}^2}$$

For ε^0 term, we have

$$\varepsilon^0: \frac{\mathrm{d}^2u_0}{\mathrm{d}\bar{t}^2} + \omega^2u_0 = 0, \qquad u_0(0) = a, \qquad \frac{\mathrm{d}u_0}{\mathrm{d}\bar{t}}(0) = 0, \qquad u_0(\bar{t}) = a\cos(\omega\bar{t})$$

For ε^1 term, we have

$$\varepsilon^1: \frac{\mathrm{d}^2u_1}{\mathrm{d}\bar{t}^2} + \omega^2u_1 = u_0^3 - 2f_1'\frac{\mathrm{d}^2u_0}{\mathrm{d}\bar{t}^2} = a^3\cos^3(\omega\bar{t}) + 2f_1'(t)\omega^2a\cos(\omega\bar{t})$$

To suppress the secular term, the coefficient of $\cos(\omega\bar{t})$ should be zero, which gives

$$\frac{3a^3}{4} + 2f_1'(t)\omega^2a = 0, \qquad f_1'(t) = -\frac{3a^2}{8\omega^2}, \qquad f_1(t) = -\frac{3a^2}{8\omega^2}t$$

Therefore, the asymptotic solution is

$$u(t;\varepsilon) \sim a\cos\left[\omega t\left(1 - \varepsilon\frac{3a^2}{8\omega^2} + \cdots\right)\right], \qquad \varepsilon \to 0$$

Method of multiple scales 2: Damped van der Pol equation

$$u''(t) + u = \varepsilon(1 - u^2)u'(t) - \varepsilon u^3, \qquad u(0;\varepsilon) = a, \qquad u'(0;\varepsilon) = b$$

We can look for an asymptotic solution in the form

$$u(t;\varepsilon) \sim \sum_{n=0}^{\infty}\varepsilon^n u_n(t^*,\tau), \qquad \tau = \varepsilon t, \qquad t^* = t(1 + w_2\varepsilon^2 + \cdots)$$

The derivative operator becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (1 + w_2\varepsilon^2 + \cdots)\frac{\partial u}{\partial t^*} + \varepsilon\frac{\partial u}{\partial\tau}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = (1 + 2w_2\varepsilon^2 + \cdots)\frac{\partial^2u}{\partial t^{*2}} + 2\varepsilon(1 + w_2\varepsilon^2 + \cdots)\frac{\partial^2u}{\partial t^*\partial\tau} + \varepsilon^2\frac{\partial^2u}{\partial\tau^2}$$

For ε^0 term, we have

$$\varepsilon^0: \frac{\partial^2u_0}{\partial t^{*2}} + u_0 = 0, \qquad u_0(0,0) = a, \qquad \frac{\mathrm{d}u_0}{\mathrm{d}\bar{t}}(0,0) = b$$

The leading order term in the solution is given as

$$u_0(t^*,\tau) = A_0(\tau)\cos t^* + B_0(\tau)\sin t^*, \qquad A_0(0) = a, \qquad B_0(0) = b$$

For ε^1 term, we have

$$\varepsilon^1: \frac{\partial^2u_1}{\partial t^{*2}} + u_1 = -2\frac{\partial^2u}{\partial t^*\partial\tau}(1 - u_0^2)\frac{\partial u_0}{\partial t^*} - u_0^3$$

The coefficients of the secular terms need to be zero, which leads to

$$\sin t^*: \ 2A_0' - A_0 + \frac{1}{4}(A_0^2 + B_0^2)(A_0 - 3B_0) = 0$$

$$\cos t^*: 2B_0' - B_0 + \frac{1}{4}(A_0^2 + B_0^2)(3A_0 + B_0) = 0$$

We still have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(A_0^2+B_0^2)=(A_0^2+B_0^2)\left[1-\frac{1}{4}(A_0^2+B_0^2)\right]$$

This inspires the following transformation

$$A_0(\tau)=R(\tau)\cos\phi(\tau)\,,\qquad B_0(\tau)=R(\tau)\sin\phi(\tau)$$

The ODE system then becomes

$$(R^2)'=R^2\left(1-\frac{R^2}{4}\right),\qquad \phi'=-\frac{3}{8}R^2,\qquad R(0)=\sqrt{a^2+b^2},\qquad \tan\phi(0)=\frac{b}{a}$$

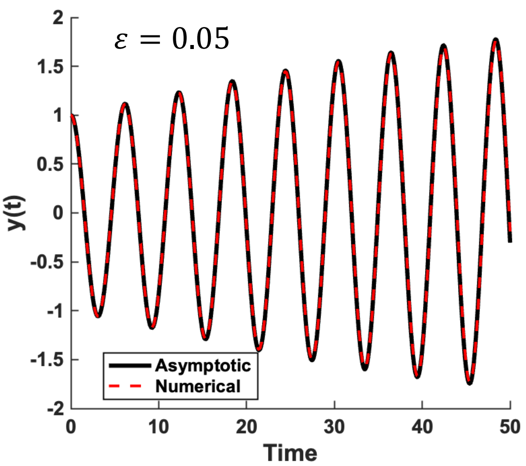
Hence we obtain

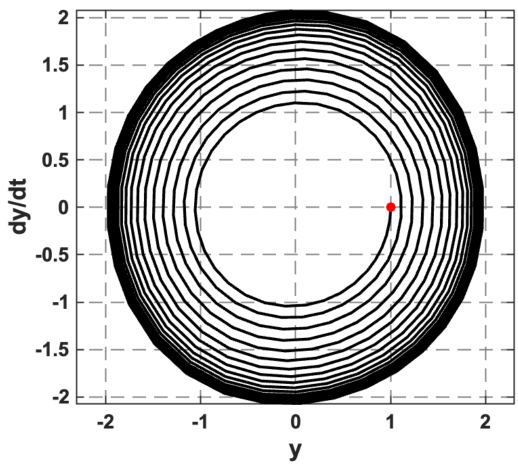
$$R(\tau)=\frac{2}{\sqrt{1+Ce^{-\tau}}},\qquad \phi(\tau)=\arctan\frac{b}{a}-\frac{3}{2}\ln\left(\frac{e^{\tau}+C}{1+C}\right),\qquad C=\frac{4}{a^2+b^2}-1$$

The leading order term of the asymptotic solution is

$$u(t;\varepsilon)\sim\frac{2}{\sqrt{1+Ce^{-\varepsilon t}}}\cos\big(t-\phi(0)\big)\,,\qquad \varepsilon\to 0$$

As $t\to\infty$, we have $R(t)\to 2$ indicating a limit cycle.





Asymptotic Analysis of Differential Equations (3): WKBJ method

For $x \in I = [\alpha, \beta] \subseteq \mathbb{R}$ and $y \in \Omega \subseteq \mathbb{R}^n$, consider the following ODE with respect to a large parameter λ given as

$$y''(x) + f(x;\lambda)\, y(x) = 0, \qquad \lambda \to +\infty$$

The function $f(x;\lambda)$ has the asymptotic expansion

$$f(x;\lambda) \sim \lambda^2 \sum_{n\geq 0} f_n(x) a_n(\lambda), \qquad \lambda \to +\infty, \qquad x \in I$$

This method originates from the analysis of Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi, \qquad \psi'' + \frac{2m(V-E)}{\hbar^2}\psi = 0$$

The classical limit corresponds to $\hbar \to 0^+$ ($\lambda = \hbar^{-1} \to +\infty$). The issue of this problem lies in the function $f(x;\lambda) \to \infty$ as $\lambda \to +\infty$. Note that

$$\frac{y''}{f} + y = 0 \qquad \Longrightarrow \qquad y = 0 \quad \text{when} \quad \lambda \to +\infty$$

We cannot obtain a useful solution from ε^0 term, since $\varepsilon = \lambda^{-1} \to 0^+$ is in the highest-order derivative term, unlike the ODE with parameters analyzed by previous methods.

- WKBJ method (7.2)

We first transform the ODE into the **Riccati equation**

$$u = (\ln y)' = \frac{y'}{y}, \qquad u' + u^2 + f = 0$$

From the solution of the Riccati equation, the solution of the original equation is

$$y = \exp\left(\int_{x_0}^{x} u(s;\lambda)\, \mathrm{d}s\right)$$

Consider the asymptotic series

$$u(x;\lambda) \sim \sum_{n\geq 0} u_n(x) b_n(\lambda), \qquad f(x;\lambda) \sim \lambda^2 \sum_{n\geq 0} f_n(x) a_n(\lambda), \qquad \lambda \to +\infty$$

This implies the following constraints

$$a_0(\lambda) = 1, \qquad a_{n+1}(\lambda) = o\big(a_n(\lambda)\big), \qquad b_{n+1}(\lambda) = o\big(b_n(\lambda)\big), \qquad \lambda \to +\infty$$

The Riccati equation becomes

$$[u_0(x)b_0(\lambda) + u_1(x)b_1(\lambda) + \cdots]' + [u_0(x)b_0(\lambda) + u_1(x)b_1(\lambda) + \cdots]^2 + \lambda^2[f_0(x)a_0(\lambda) + f_1(x)a_1(\lambda) + \cdots] = 0$$

The leading order terms are

$$u_0'(x)b_0(\lambda) + u_0^2(x)b_0^2(\lambda) + \lambda^2 f_0(x) = o\big(b_0(\lambda)\big) + o\big(b_0^2(\lambda)\big) + o(\lambda^2)$$

We analyze the dominant balance among these three terms, and we need to set

$$b_0(\lambda) = \lambda, \qquad u_0^2(x) + f_0(x) = o(\lambda^2), \qquad u_0(x) = \pm\sqrt{-f_0(x)}$$

The **turning points** at which $f_0(x) = 0$ govern the behavior of the solution in different regimes. The next order terms give

$$u_0'(x)\lambda + 2u_0(x)u_1(x)\lambda b_1(\lambda) + \lambda^2 f_1(x)a_1(\lambda) = o(\lambda) + o\big(\lambda b_1(\lambda)\big) + o\big(\lambda^2 a_1(\lambda)\big)$$

There are several cases depending on the order of $a_1(\lambda)$:

- Case 1: Dominant balance of term I and II (special $C = 0$ of Case 3)

$$\lambda^2 a_1(\lambda) = o(\lambda), \qquad a_1(\lambda) = o\left(\frac{1}{\lambda}\right), \qquad b_1(\lambda) = 1, \qquad u_1(x) = -\frac{u_0'}{2u_0}$$

- Case 2: Dominant balance of term II and III

$$\lambda = o\big(\lambda^2 a_1(\lambda)\big), \qquad b_1(\lambda) = \lambda a_1(\lambda), \qquad u_1(x) = -\frac{f_1}{2u_0}$$

- Case 3: Dominant balance of all three terms

$$\lim_{\lambda\to+\infty} \lambda a_1(\lambda) = C, \qquad b_1(\lambda) = 1, \qquad u_1(x) = -\frac{u_0' + Cf_1}{2u_0}$$

This process ends when we reach $b_N(\lambda) = O(1)$, and this gives

$$u(x;\lambda) \sim \sum_{n=0}^{N} u_n(x)b_n(\lambda) + o(1), \qquad \lambda \to +\infty$$

$$y(x;\lambda) \sim \exp\left(\sum_{n=0}^{N} b_n(\lambda)\int_{x_0}^{x} u_n(s)\,\mathrm{d}s\right)\big(1 + o(1)\big), \qquad \lambda \to +\infty$$

Usually, we study the case with $a_n(\lambda) = \lambda^{-n}$, from which we choose $u(x;\lambda)$ as

$$u(x;\lambda) \sim \lambda \sum_{n\geq 0} u_n(x)\lambda^{-n}$$

The Riccati equation becomes

$$\sum_{n\geq 0} u_n'(x)\lambda^{-n-1} + \left(\sum_{k\geq 0} u_k(x)\lambda^{-k}\right)\left(\sum_{l\geq 0} u_l(x)\lambda^{-l}\right) + \sum_{n\geq 0} f_n(x)\lambda^{-n} = 0$$

For each order of λ, we have

$$\lambda^0\colon\ u_0^2 + f_0 = 0, \qquad u_0 = \pm\sqrt{-f_0}$$

$$\lambda^{-n}\colon\ u_{n-1}' + \sum_{k=0}^{n} u_k u_{n-k} + f_n = 0, \qquad u_n = -\frac{1}{2u_0}\left(u_{n-1}' + \sum_{k=1}^{n-1} u_k u_{n-k} + f_n\right)$$

Consider $x_0 \in (\alpha,\beta)$ such that $f(x_0) \neq 0$, and assume that we can further find $\delta > 0$ such that $f_0(x) > M$ or $f_0(x) < -M$ when $x_0 \in B(x_0,\delta)$. We also assume $f_n \in C^\infty$, and we have

$$u(x;\lambda) \sim \lambda u_0(x) + u_1(x) + O(\lambda^{-1}), \qquad \lambda \to +\infty$$

The first two terms are given as

$$u_0(x) = \pm\sqrt{-f_0}, \qquad u_1(x) = -\frac{u_0' + f_1}{2u_0} = -\frac{1}{4}\frac{f_0'}{f_0} - \frac{f_1}{2u_0}$$

Therefore, we have the solutions

$$u^{\pm}(x;\lambda) = \pm\lambda\sqrt{-f_0} - \frac{1}{4}\frac{f_0'}{f_0} \pm \frac{f_1}{2\sqrt{-f_0}}$$

$$y^{\pm}(x;\lambda) = f_0^{-\frac{1}{4}} \exp\left(\pm\int_{x_0}^{x} \lambda\sqrt{-f_0}\,\mathrm{d}s \pm \frac{1}{2}\int_{x_0}^{x} \frac{f_1}{\sqrt{-f_0}}\,\mathrm{d}s\right) \cdot \big(1 + o(1)\big)$$

More specifically, depending on the sign of $f_0(x)$ on $B(x_0, \delta)$, we have

$$y^{\pm}(x;\lambda) = f_0^{-\frac{1}{4}} \exp\left(\pm i\lambda\int_{x_0}^{x} \sqrt{f_0}\,\mathrm{d}s \pm \frac{i}{2}\int_{x_0}^{x} \frac{f_1}{\sqrt{f_0}}\,\mathrm{d}s\right) \cdot \big(1 + o(1)\big), \qquad f_0(x_0) > 0$$

$$y^{\pm}(x;\lambda) = |f_0|^{-\frac{1}{4}} \exp\left(\pm\lambda\int_{x_0}^{x} \sqrt{|f_0|}\,\mathrm{d}s \mp \frac{1}{2}\int_{x_0}^{x} \frac{f_1}{\sqrt{|f_0|}}\,\mathrm{d}s\right) \cdot \big(1 + o(1)\big), \qquad f_0(x_0) < 0$$

Example

$$y''(x) + [\lambda^2 + \varepsilon\mu(x)]y(x) = 0, \qquad \lambda \to +\infty$$

This describes the propagation of light in a medium with spatial variation in the refractive index given by $\varepsilon\mu(x)$. For this equation, we have

$$f_0 = 1, \qquad f_1 = 0, \qquad f_2 = \varepsilon\mu(x), \qquad f_n = 0, \qquad n \geq 3$$

Directly from the WKBJ method, we can obtain

$$u_0^{\pm}(x) = \pm i, \qquad u_1^{\pm}(x) = 0, \qquad u_2^{\pm}(x) = \pm\frac{i\varepsilon}{2}\mu(x)$$

Since $f_0 > 0$, the solution is

$$y^{\pm}(x;\lambda) = \exp\left(\pm i\lambda(x - x_0) \pm \frac{i\varepsilon}{2\lambda}\int_{x_0}^{x} \mu(s)\,\mathrm{d}s\right) \cdot \big(1 + o(\lambda^{-1})\big), \qquad \lambda \to +\infty$$

The dominant term is a high-frequency oscillation.

Consistency of WKBJ asymptotic series

We first study the case with $f_0(x_0) > 0$, which gives

$$u_0^{\pm}(x) = \pm i\sqrt{f_0(x)}, \qquad u_n^{\pm}(x) = -\frac{1}{2u_0^{\pm}}\left({u_{n-1}^{\pm}}' + \sum_{k=1}^{n-1} u_k^{\pm}u_{n-k}^{\pm} + f_n\right)$$

Denote the exact solution as $u(x;\lambda)$. We want to know if there exists $\delta > 0$ such that

$$\frac{u(x;\lambda)-\lambda\sum_{n=0}^{N}u_n(x)\lambda^{-n}}{\lambda^{-N-1}} \rightrightarrows 0, \qquad N\to\infty, \qquad x\in B(x_0,\delta)$$

Denote the partial sum as $u_N(x;\lambda)$ and the difference as $\Delta_N(x;\lambda)=u(x;\lambda)-u_N(x;\lambda)$. The initial value of the error at $x=x_0$ satisfies

$$U_N(\lambda)=\Delta_N(x_0;\lambda)=O(\lambda^{-N}), \qquad \lambda\to+\infty$$

With $u=u_N+\Delta_N$, substituting it into the Riccati equation leads to

$$\Delta_N'+u_N'+\Delta_N^2+2u_N\Delta_N+u_N^2+f=0, \qquad \Delta_N'+\Delta_N^2+2\lambda u_0\Delta_N+A_N\Delta_N=B_N$$

We organize some terms into A_N and B_N with the following behaviors

$$A_N(x;\lambda)=2\sum_{n=1}^{N}u_n\lambda^{-n+1}=O(1), \qquad B_N(x;\lambda)=u_N'+u_N^2+f=O(\lambda^{-N+1})$$

Introduce an exponential integrating factor

$$E(x,y;\lambda)=\exp\left(2\lambda\int_y^x u_0(s)\,\mathrm{d}s\right)=\exp\left(2i\lambda\int_y^x\sqrt{f_0(s)}\,\mathrm{d}s\right), \qquad \frac{\partial E}{\partial x}=2\lambda u_0(x)E$$

Apply it to the Riccati equation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[E(x,x_0;\lambda)\Delta_N(x;\lambda)]=2\lambda u_0\Delta_N E+\Delta_N' E=(B_N-A_N\Delta_N-\Delta_N^2)E$$

Integrate both sides, and with the initial values we have

$$E(x,x_0;\lambda)\Delta_N(x;\lambda)-U_N(\lambda)=\int_{x_0}^{x}E(s,x_0;\lambda)(B_N-A_N\Delta_N-\Delta_N^2)\,\mathrm{d}s$$

Since $E(y,x;\lambda)=E^{-1}(x,y;\lambda)$, this leads to

$$\Delta_N(x;\lambda)=T_N[\Delta_N](x;\lambda)=U_N(\lambda)E(x_0,x;\lambda)+\int_{x_0}^{x}E(s,x;\lambda)(B_N-A_N\Delta_N-\Delta_N^2)\,\mathrm{d}s$$

The nonlinear integral operator is denoted as T_N. The above expression implies that Δ_N is a fixed point of T_N. For two functions $w_1,w_2\in\{f\in C^0\ |\|f\|=\sup|f(x)|\le M\}$, we have

$$\|T_N[w_1]-T_N[w_2]\|\le\left|\int_{x_0}^{x}(|A_N|+|w_1|+|w_2|)\,\mathrm{d}s\right|\|w_2-w_1\|\le C_1\delta\|w_2-w_1\|$$

We can choose δ such that $\|T_N[w_1]-T_N[w_2]\|\le\|w_2-w_1\|$, which is a contraction mapping. Eventually, we have

$$\|\Delta_N\|=\|T_N[\Delta_N]\|\le\frac{C_2}{\lambda^N}=O(\lambda^{-N})$$

This proves that if initially we have $O(\lambda^{-N})$ error at $x=x_0$, then it holds uniformly over (α,β).

In terms of the case with $f_0(x_0)<0$, for the $+$ solution we have

$$u_0^+(x)=\sqrt{|f_0(x)|}$$

The integrating factor now becomes

$$E^{+}(x,y;\lambda) = \exp\left(2\lambda\int_{y}^{x}\sqrt{|f_0(s)|}\,\mathrm{d}s\right)$$

When $\lambda \to +\infty$, it is an exponential growth ($x > y$) or decay ($x < y$). In this case, $y^{+}(x;\lambda)$ is consistent only in $[x_0, x_0+\delta)$. Similarly, y^{-} is consistent only in $(x_0-\delta, x_0]$. The validity of WKBJ asymptotics is in the direction of exponential growth.

Turning points

Consider the following example

$$f(x;\lambda) = \lambda^2 x, \qquad y'' + \lambda^2 xy = 0$$

The ODE can be transformed into

$$z = -\lambda^{2/3}x, \qquad Y'(z) - zY(z) = 0$$

This is the Airy equation, from which we can write down the general solution as

$$Y(z) = C_1\mathrm{Ai}(z) + C_2\mathrm{Bi}(z)$$

For $x < 0$, $f(x) < 0$, when $\lambda \to +\infty$ we have $z \to +\infty$

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}}\frac{1}{z^{1/4}}e^{-\frac{2}{3}z^{3/2}}\left(1 + O\left(z^{-3/2}\right)\right)$$

$$\mathrm{Bi}(z) = \frac{1}{\sqrt{\pi}}\frac{1}{z^{1/4}}e^{\frac{2}{3}z^{3/2}}\left(1 + O\left(z^{-3/2}\right)\right), \qquad z \to +\infty$$

For $x > 0$, $f(x) > 0$, when $\lambda \to +\infty$ we have $z \to -\infty$

$$\mathrm{Ai}(z) = \frac{1}{\sqrt{\pi}}\frac{1}{|z|^{1/4}}\left[\sin\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) + O\left(|z|^{-3/2}\right)\right]$$

$$\mathrm{Bi}(z) = \frac{1}{\sqrt{\pi}}\frac{1}{|z|^{1/4}}\left[\cos\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right) + O\left(|z|^{-3/2}\right)\right], \qquad z \to -\infty$$

Near the turning point, the transition from exponential to oscillatory behaviors is connected by the Airy function. We can define three regimes: left (y_L, exponential), middle (y_M, Airy) and right (y_R, oscillatory). The connection problem is to find a proper y_R given the solution y_L, such that there exists y_M to form a smooth solution near the turning point x_*.

Consider another example

$$f(x;\lambda) = \lambda^2\mathrm{sgn}(x), \qquad y''(x) + f(x;\lambda)\,y(x) = 0$$

We can obtain the exact general solution as

$$y = Ae^{\lambda x} + Be^{-\lambda x}, \qquad x < 0, \qquad y = Ce^{i\lambda x} + De^{-i\lambda x}, \qquad x > 0$$

The connection conditions can be chosen as $y(0^-) = y(0^+)$ and $y'(0^-) = y'(0^+)$. This gives

$$C = \frac{1-i}{2}A + \frac{1+i}{2}B, \qquad D = \frac{1+i}{2}A + \frac{1-i}{2}B$$

For asymptotic analysis, we first need to choose a solution in $x < 0$ such that

$$y_L^+ = e^{\lambda x}\big(1 + o(1)\big), \qquad x < 0, \qquad \lambda \to +\infty$$

In this case, the connection problem is well-posed and we obtain

$$A = 1, \qquad B = 0, \qquad C = \frac{1-i}{2}, \qquad D = \frac{1+i}{2}$$

However, if we choose the other solution in $x < 0$ such that

$$y_L^- = e^{-\lambda x}\big(1 + o(1)\big), \qquad x < 0, \qquad \lambda \to +\infty$$

Since $e^{\lambda x} = e^{-\lambda x} o(1)$ for $x < 0$, the coefficient A is arbitrary and thus we cannot determine a unique oscillatory solution in $x > 0$ to form the connection. Vice versa, if we choose a solution in $x > 0$ such that

$$y_R = Ce^{i\lambda x} + De^{-i\lambda x} + o(1), \qquad x > 0, \qquad \lambda \to +\infty$$

The connection conditions give

$$A = \frac{1+i}{2}C + \frac{1-i}{2}D, \qquad B = \frac{1-i}{2}C + \frac{1+i}{2}D$$

The asymptotic behavior for the solution in $x < 0$ is

$$y_L = Ae^{\lambda x} + Be^{-\lambda x} = Be^{-\lambda x}\big(1 + o(1)\big), \qquad x < 0, \qquad \lambda \to +\infty$$

If $B = 0$, then we need to refer to higher order terms to pose the connection problem.

We assume that $I = (\alpha, \beta)$ only has one turning point x_* such that with $f_0'(x_*) = \nu^2 > 0$. The functions are smooth with $f_n \in C^\infty(I)$. For $f_0(x)$, near the turning point we have

$$f_0(x) = \nu^2(x - x_*) + o(x - x_*)$$

We want to find $\delta(\lambda)$ such that there exists a consistent asymptotic solution within the region

$$x \in \big(x_1, x_* - \delta(\lambda)\big) \cup (x_* + \delta(\lambda), x_2)$$

Based on WKBJ results, we first obtain

$$u_0(x) = \pm\sqrt{-\nu^2(x - x_*) + o(x - x_*)} = O\left(\sqrt{|x - x_*|}\right)$$

Specifically, we have

$$u_0 = \pm\nu\sqrt{|x - x_*|} + o\left(\sqrt{|x - x_*|}\right), \qquad x < x_*$$

$$u_0 = \pm i\nu\sqrt{x - x_*} + o\big(\sqrt{x - x_*}\big), \qquad x > x_*$$

Since f_n is finite, by induction we can further estimate

$$u_1(x) = -\frac{u_0' + f_1}{2u_0} = O(|x - x_*|^{-1})$$

$$u_n(x) = -\frac{1}{2u_0}\left(u_{n-1}' + \sum_{k=1}^{n-1} u_k u_{n-k} + f_n\right) = O\left(|x - x_*|^{\frac{1-3n}{2}}\right)$$

To obtain a valid asymptotic series, we require

$$\lim_{\lambda\to+\infty}\frac{u_{n+1}\lambda^{-n}}{u_n\lambda^{-(n-1)}}=\frac{|x-x_*|^{\frac{1}{2}-\frac{3}{2}(n+1)}\lambda^{-n}}{|x-x_*|^{\frac{1}{2}-\frac{3}{2}n}\lambda^{-n+1}}=\lambda^{-1}|x-x_*|^{-\frac{3}{2}}=0$$

This leads to the condition on $\delta(\lambda)$ as

$$|x-x_*|^{-\frac{3}{2}}=o(\lambda),\qquad \delta(\lambda)=\lambda^{-p},\qquad 0<p<\frac{2}{3}$$

In the outer region, it can be shown that the consistent solution exists for $0<p<2/3$.

For $x\in(x_1,x_*-\lambda^{-p})$, we have

$$y_L^+(x;\lambda)=|f_0(x)|^{-\frac{1}{4}}\exp\left(\lambda\int_{x_*}^{x}\sqrt{|f_0(s)|}\,\mathrm{d}s-\frac{1}{2}\int_{x_*}^{x}\frac{f_1(s)}{\sqrt{|f_0(s)|}}\mathrm{d}s\right)\left(1+O\left(\lambda^{\frac{3p}{2}-1}\right)\right)$$

For $x\in(x_*+\lambda^{-p},x_2)$, we have

$$y_R^\pm(x;\lambda)=f_0(x)^{-\frac{1}{4}}\exp\left(\pm i\lambda\int_{x_*}^{x}\sqrt{f_0(s)}\,\mathrm{d}s\pm\frac{i}{2}\int_{x_*}^{x}\frac{f_1(s)}{\sqrt{f_0(s)}}\mathrm{d}s\right)\left(1+O\left(\lambda^{\frac{3p}{2}-1}\right)\right)$$

Across the turning point for $x\in(x_*-\delta(\lambda),x_*+\delta(\lambda))$, we need to directly solve the equation. We assume that for sufficiently large λ, $f(x;\lambda)$ only have one zero point $x_*(\lambda)$. As $\lambda\to+\infty$, $x_*(\lambda)$ becomes the turning point x_*. For sufficiently small $|x-x_*|$, we uniformly have

$$f_0(x;\lambda)=\nu^2\big(x-x_*(\lambda)\big)+o\big(x-x_*(\lambda)\big)$$

$$f(x;\lambda)=\lambda^2\nu^2\big(x-x_*(\lambda)\big)\Big(1+\big(x-x_*(\lambda)\big)h(x;\lambda)\Big),\qquad h=O(1)$$

With the same transformation

$$x=x_*(\lambda)-\alpha z,\qquad \alpha=(\lambda\nu)^{-2/3},\qquad Y(z;\lambda)=y(x_*(\lambda)-\alpha z;\lambda)$$

The original ODE is converted into the Airy equation

$$Y''-zY=\lambda^{-2/3}z^2g(z;\lambda)Y,\qquad g=O(1),\qquad z\to 0$$

Consider the general solution satisfying

$$Y(z;\lambda)=a(z;\lambda)\,\mathrm{Ai}(z)+b(z;\lambda)\,\mathrm{Bi}(z),\qquad Y'(z;\lambda)=a(z;\lambda)\,\mathrm{Ai}'(z)+b(z;\lambda)\,\mathrm{Bi}'(z)$$

This implies that we choose the coefficients under the following constraint

$$a'(z;\lambda)\,\mathrm{Ai}(z)+b'(z;\lambda)\,\mathrm{Bi}(z)=0$$

Now the Airy equation becomes

$$a'(z;\lambda)\,\mathrm{Ai}'(z)+b'(z;\lambda)\,\mathrm{Bi}'(z)=\lambda^{-2/3}z^2g(z;\lambda)\big(a(z;\lambda)\,\mathrm{Ai}(z)+b(z;\lambda)\,\mathrm{Bi}(z)\big)$$

Using the asymptotic behaviors of the Airy functions and their derivatives, the determinant of the Wronskian and its inverse are obtained as

$$\det W=\mathrm{Ai}(z)\,\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\,\mathrm{Bi}(z)=\frac{1}{\pi},\qquad W^{-1}=\frac{1}{\det W}\begin{pmatrix}\mathrm{Bi}'(z) & -\mathrm{Bi}(z)\\ -\mathrm{Ai}'(z) & \mathrm{Ai}(z)\end{pmatrix}$$

These lead to an ODE system for $a(z;\lambda)$ and $b(z;\lambda)$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = -\pi\lambda^{-\frac{2}{3}} z^2 g \begin{pmatrix} \mathrm{Ai}(z)\,\mathrm{Bi}(z) & [\mathrm{Bi}(z)]^2 \\ -[\mathrm{Ai}(z)]^2 & -\mathrm{Ai}(z)\,\mathrm{Bi}(z) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

With initial conditions at $x - x_*(\lambda) = \pm\lambda^{-p_1}$ given by the outer solution, we want to show that the coefficients a, b are nearly constant when $\lambda \to +\infty$, as indicated by the ODE system.

$$x - x_*(\lambda) = -\alpha z = \mp\lambda^{-p_1}, \qquad z = \pm\nu^{2/3}\lambda^{2/3-p_1} = \pm C\lambda^q$$

The initial values at the left endpoint can be written as

$$a(C\lambda^q;\lambda) = 1 + \delta a, \qquad b(C\lambda^q;\lambda) = 0 + \tilde{C}\delta b, \qquad \delta a, \delta b = o(1)$$

Integrate the ODE from $C\lambda^q$ to an arbitrary z with $|z| < C\lambda^q$

$$\begin{pmatrix} a(z;\lambda) - 1 - \delta a \\ b(z;\lambda) - \tilde{C}\delta b \end{pmatrix} = -\pi\lambda^{-\frac{2}{3}} \int_{C\lambda^q}^{z} \zeta^2 g(\zeta;\lambda) \begin{pmatrix} \mathrm{Ai}(\zeta)\,\mathrm{Bi}(\zeta) & [\mathrm{Bi}(\zeta)]^2 \\ -[\mathrm{Ai}(\zeta)]^2 & -\mathrm{Ai}(\zeta)\,\mathrm{Bi}(\zeta) \end{pmatrix} \begin{pmatrix} a(\zeta) \\ b(\zeta) \end{pmatrix} \mathrm{d}\zeta$$

From this integral equation, it can be shown that

$$\|\tilde{a}\| = \|a(z;\lambda) - 1\| \le |\delta a| + C_1\lambda^{\frac{5q}{2}-\frac{2}{3}}\left(\|\tilde{a}\| + \|\tilde{b}\| + 1\right)$$

$$\|\tilde{b}\| = \|W(z)b(z;\lambda)\| \le |\delta b| + C_1\lambda^{\frac{5q}{2}-\frac{2}{3}}\left(\|\tilde{a}\| + \|\tilde{b}\| + 1\right)$$

The function $W(z)$, which describes the order of $[\mathrm{Bi}(z)]^2$, is defined by

$$W(z) = e^{\frac{4}{3}z^{3/2}}, \qquad z > 0, \qquad W(z) = 1, \qquad z \le 0$$

For the norm to be bounded by a finite value when $\lambda \to +\infty$, we require

$$\frac{5q}{2} - \frac{2}{3} < 0, \qquad q = \frac{2}{3} - p_1 < \frac{4}{15}, \qquad p_1 > \frac{2}{5}$$

Therefore, the **overlap domain** between the inner and outer regimes is

$$\lambda^{-2/3} < |x - x_*| < \lambda^{-2/5}$$

Suppose that $x_L = x_*(\lambda) - \lambda^{-p}$ with $2/5 < p < 2/3$, as $\lambda \to +\infty$ it can be shown that

$$|f_0(x)|^{-\frac{1}{4}} = \nu^{-\frac{1}{2}}\lambda^{\frac{p}{4}}\left(1 + O(\lambda^{-p}) + O(\lambda^{p-1})\right), \qquad \int_{x_*}^{x} \frac{f_1(s)}{\sqrt{|f_0(s)|}}\mathrm{d}s = O\left(\lambda^{-\frac{p}{2}}\right)$$

$$\lambda\int_{x_*}^{x}\sqrt{|f_0(s)|}\,\mathrm{d}s = -\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}}\left(1 + O(\lambda^{-p}) + O(\lambda^{p-1})\right)$$

With these results, we have

$$y_L^+(x_L;\lambda) = \nu^{-\frac{1}{2}}\lambda^{\frac{p}{4}}\exp\left(-\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + O\left(\lambda^{-\frac{p}{2}}\right) + O\left(\lambda^{\frac{3p}{2}-1}\right) + O\left(\lambda^{1-\frac{5p}{2}}\right)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}y_L^+(x_L;\lambda) = \nu^{\frac{1}{2}}\lambda^{1-\frac{p}{4}}\exp\left(-\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + O\left(\lambda^{-\frac{p}{2}}\right) + O\left(\lambda^{\frac{3p}{2}-1}\right) + O\left(\lambda^{1-\frac{5p}{2}}\right)\right)$$

In the overlap domain, y_L^+ can be represented by a linear combination of the Airy functions.

From previous analysis, the coefficients are given as

$$a(z;\lambda) = \pi \mathrm{Bi}'(z) Y(z;\lambda) - \pi \mathrm{Bi}(z) \frac{\mathrm{d}Y}{\mathrm{d}z}(z;\lambda)$$

$$b(z;\lambda) = \pi \mathrm{Ai}(z) \frac{\mathrm{d}Y}{\mathrm{d}z}(z;\lambda) - \pi \mathrm{Ai}'(z) Y(z;\lambda)$$

For the connection problem, we have

$$Y(z;\lambda) = y_L^+(C\lambda^q;\lambda), \qquad x_L - x_*(\lambda) = -\alpha z_L = -\lambda^{-p_1}, \qquad z_L = \nu^{2/3}\lambda^{2/3-p_1} = C\lambda^q$$

At the left endpoint of the overlap domain, we obtain

$$a(C\lambda^q;\lambda) = 2\sqrt{\pi}\,\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\,(1+\delta a), \qquad b(C\lambda^q;\lambda) = \frac{\sqrt{\pi}}{2}\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\exp\left(-\frac{4}{3}C^{\frac{3}{2}}\lambda^{\frac{3q}{2}}\right)(1+\delta b)$$

Hence, in the neighborhood $|x - x_*(\lambda)| \le \lambda^{-p_1}$, or equivalently $|z| \le C\lambda^q$, we have

$$y_L^+(x;\lambda) = Y_L^+(z;\lambda) = 2\sqrt{\pi}\,\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\left[1 + E_p(\lambda)\right]\mathrm{Ai}(z)$$

$$+2\sqrt{\pi}\,\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\,W^{-1}(z)\,E_p(\lambda)\,\mathrm{Bi}(z)$$

For the asymptotic expression of $\mathrm{d}Y_L^+/\mathrm{d}z$, just add a derivative to the Airy functions. The error term $E_p(\lambda)$ represents a possibly different function satisfying

$$E_p(\lambda) = O\left(\lambda^{-p/2}\right) + O\left(\lambda^{3p/2-1}\right) + O\left(\lambda^{1-5p/2}\right), \qquad \lambda \to +\infty$$

The optimal error is obtained when $p = 1/2$, which leads to an estimate of $O(\lambda^{-1/4})$.

We also need to represent y_L^+ as a linear combination of the oscillatory solutions at the right endpoint x_R. Specifically, take $x_R = x_*(\lambda) + \lambda^{-p_1}$ (equivalently $z_R = -C\lambda^q$) and we have

$$y_L^+(x_R;\lambda) = 2\nu^{-\frac{1}{2}}\,\lambda^{\frac{p}{4}}\left[\sin\left(\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}} + \frac{\pi}{4}\right) + E_p(\lambda)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}y_L^+(x_R;\lambda) = 2\nu^{\frac{1}{2}}\,\lambda^{1-\frac{p}{4}}\left[\cos\left(\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}} + \frac{\pi}{4}\right) + E_p(\lambda)\right]$$

The oscillatory solutions are given as

$$y_R^\pm(x_R;\lambda) = \nu^{-\frac{1}{2}}\,\lambda^{\frac{p}{4}}\exp\left(\pm\frac{2i}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + E_p(\lambda)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}y_R^\pm(x_R;\lambda) = \pm i\nu^{\frac{1}{2}}\,\lambda^{1-\frac{p}{4}}\exp\left(\pm\frac{2i}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + E_p(\lambda)\right)$$

Hence, we have

$$y_L^+(x;\lambda) = \left(e^{-\pi i/4} + E_p(\lambda)\right)y_R^+(x;\lambda) + \left(e^{\pi i/4} + E_p(\lambda)\right)y_R^-(x;\lambda)$$

This is similar to the result of a previous example where $f(x;\lambda) = \lambda^2\mathrm{sgn}(x)$.

For $f_0(x)$ with the following form at the turning point x_*

$$f_0(x) = (x - x_*)^m g(x)$$

The procedure remains similar, but instead of Airy functions, we need others for connection.

➢ Langer transformation (7.2.5)

For simplicity, consider $f(x;\lambda)=\lambda^2 f_0(x)$ that only contains the leading order term

$$y''(x)+\lambda^2 f_0(x)y(x)=0,\qquad \lambda\to+\infty$$

We consider the nonlinear transformation

$$y(x;\lambda)=a(x)v(x;\lambda),\qquad x=g(\xi),\qquad V(\xi;\lambda)=v(x;\lambda)$$

Eventually, we obtain an ODE of the form

$$V''(\xi;\lambda)-\lambda^2\xi V(\xi;\lambda)=F(\xi)V(\xi;\lambda),\qquad F(\xi)=-\left.\frac{a''(x)}{a(x)[g'(x)]^2}\right|_{x=g^{-1}(\xi)}$$

Langer's method provides a uniformly small error of $O(\lambda^{-1})$ over a fixed-size interval around the turning point, and can give more accurate information in the neighborhoods than techniques based on rescaling and matching to WKBJ formulae.

➢ Exercise

Wave equation in a medium with vertically varying property

Assume a medium with vertically varying phase speed $c(z)$. For a propagating wave solution of the form $\phi(z)e^{i(kx+ly-\omega t)}$, the wave equation becomes

$$\phi''(z)+m^2(z)\phi(z)=0,\qquad m^2(z)=\frac{\omega^2}{c^2(z)}-k^2-l^2=m_o^2\cdot\frac{m^2(z)}{m_o^2}=m_o^2\,\tilde{m}^2(z)$$

Here $\phi(z)$ can be interpreted as the mode coefficient and $m(z)$ the vertical wavenumber. We consider a reference value $m_o\gg\tilde{m}(z)$ as the parameter. Under the **high frequency limit** $\omega\to+\infty$, we also have $f(z;m_o)=m^2(z)\to+\infty$, which can be studied by the WKBJ method. We consider the following asymptotic solution

$$u(z;m_o)\sim m_o\sum_{n\geq 0}u_n(z)m_o^{-n},\qquad u(z;m_o)=\frac{\phi'}{\phi}=(\ln\phi)'$$

We recognize the expansion of $f(z;m_o)$ as

$$f(z;m_o)=m_o^2\sum_{n\geq 0}f_n(z)m_o^{-n},\qquad f_0(z)=\tilde{m}^2(z)=\frac{m^2(z)}{m_o^2}$$

We always have $f_0(z)>0$. Hence, the WKBJ result gives

$$u_0(z)=\sqrt{-f_0}=i\tilde{m}(z),\qquad u_1(z)=-\frac{f_0'}{4f_0}-\frac{f_1}{2u_0}=-\frac{f_0'}{4f_0}$$

The positive root is chosen for $u_0(z)$. The solution for $\phi(z)$ is

$$\phi(z)\sim f_0^{-\frac{1}{4}}\exp\left(im_0\int_{z_0}^{z}\sqrt{f_0}\,\mathrm{d}s\right)=\sqrt{\frac{m_o}{m(z)}}\exp\left(i\int_{z_0}^{z}m(s)\mathrm{d}s\right)$$

The amplitude is proportional to $m^{-1/2}$, which arises from u_1 term. The phase is accumulated along the vertical propagation path that comes from u_0 term.

Eikonal equation

Consider the 3D acoustic wave equation for pressure $\hat{p}(\boldsymbol{x},\omega)$ under the high frequency limit

$$\nabla^2\hat{p}+\frac{\omega^2}{c^2}\hat{p}=0,\qquad \omega\to+\infty$$

We assume that the density ρ is constant, and the phase speed $c(\boldsymbol{x})$ varies in space. The WKBJ solution seeks for one that consists of an oscillatory exponential factor modified by slowly varying amplitude. The asymptotic series for the solution takes the form

$$\hat{p}(\boldsymbol{x},\omega)=e^{i\omega\tau(x)}\sum_{n\geq 0}\frac{A_n(\boldsymbol{x})}{(i\omega)^n}$$

The assumption here is that only one geometrical wavefront passes through each point. If there are several wavefronts, the solution can be represented by the addition of such series. The phase factor $\omega\tau(\boldsymbol{x})$, where $\tau(\boldsymbol{x})$ denotes the travel time, can also be written as $k_0L(\boldsymbol{x})$, e.g., in optics where $L(\boldsymbol{x})$ denotes the optical path. Now consider the general case

$$\nabla^2\hat{p}+f(\boldsymbol{x};\omega)\hat{p}=0,\qquad f(\boldsymbol{x};\omega)=(i\omega)^2\sum_{n\geq 0}\frac{f_n(\boldsymbol{x})}{(i\omega)^n}$$

The equation becomes

$$(i\omega\nabla^2\tau-\omega^2|\nabla\tau|^2)\sum_{n\geq 0}\frac{A_n}{(i\omega)^n}+2i\omega\nabla\tau\cdot\sum_{n\geq 0}\frac{\nabla A_n}{(i\omega)^n}+\sum_{n\geq 0}\frac{\nabla^2 A_n}{(i\omega)^n}$$
$$+(i\omega)^2\left[\sum_{k\geq 0}\frac{f_k}{(i\omega)^k}\right]\left[\sum_{l\geq 0}\frac{A_l}{(i\omega)^l}\right]=0$$

Each order gives the following equation

$$(i\omega)^2\!:\ |\nabla\tau|^2+f_0=0,\qquad (i\omega)^1\!:\ A_0\nabla^2\tau+A_1|\nabla\tau|^2+2\nabla\tau\cdot\nabla A_0+(f_0A_1+f_1A_0)=0$$

$$(i\omega)^{2-n}\!:\ A_{n-1}\nabla^2\tau+A_n|\nabla\tau|^2+2\nabla\tau\cdot\nabla A_{n-1}+\nabla^2A_{n-2}+\sum_{k=0}^{n}f_kA_{n-k}=0,\qquad n\geq 2$$

For our specific case, we have

$$f_0(\boldsymbol{x})=-\frac{1}{c^2(\boldsymbol{x})},\qquad f_n(\boldsymbol{x})=0,\qquad n\geq 1$$

The highest order leads to the eikonal equation, with $\tau(\boldsymbol{x})$ contours being the wavefronts

$$|\nabla\tau|^2=\frac{1}{c^2(\boldsymbol{x})}$$

The next order gives the first transport equation, which is the conservation law of amplitude.

$$A_0\nabla^2\tau+2\nabla\tau\cdot\nabla A_0=0,\qquad \nabla\cdot(A_0^2\nabla\tau)=0$$

Asymptotic Analysis of Differential Equations (4): BV Problems

For $x \in I = [\alpha, \beta] \subseteq \mathbb{R}$ and $a(x), b(x) \in C[\alpha, \beta]$, consider the boundary-value problem with respect to a small parameter ε given as

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(\alpha) = A, \qquad y(\beta) = B, \qquad \varepsilon \to 0^+$$

- Existence of solutions for BVP (8.1)

For a fixed $I = [\alpha, \beta]$ with $y(\alpha) = A$ and $y(\beta) = B$, denote this problem as $\mathrm{BVP}(A, B)$.

Theorem. $\mathrm{BVP}(A, B)$ has a unique solution is equivalent to $\mathrm{BVP}(0,0)$ has a unique solution, which is the trivial solution $y(x; \lambda) = 0$.

Proof. When $\mathrm{BVP}(A, B)$ has a unique solution, it is obvious that $y = 0$ is the unique solution for $\mathrm{BVP}(0,0)$. Now consider $\mathrm{BVP}(0,0)$ has a unique trivial solution. We start from the solutions of the following initial-value problems (IVP)

$$\varepsilon y_1'' + a y_1' + b y_1 = 0, \qquad y_1(\alpha) = 0, \qquad y_1'(\alpha) = 1$$

$$\varepsilon y_2'' + a y_2' + b y_2 = 0, \qquad y_2(\beta) = 0, \qquad y_2'(\beta) = 1$$

We say that $y_1(\beta) \neq 0$ and $y_2(\alpha) \neq 0$. If not, then y_i becomes the solution of $\mathrm{BVP}(0,0)$ but with non-zero y_i' at the boundary, which contradicts the uniqueness of the trivial solution. Then we can construct the solution of $\mathrm{BVP}(A, B)$ as

$$y(x) = \frac{B}{y_1(\beta)} y_1(x) + \frac{A}{y_2(\alpha)} y_2(x)$$

This solution is unique since $\mathrm{BVP}(0,0)$ only has the trivial solution. ■

Now we only need to study when $\mathrm{BVP}(0,0)$ has a unique solution

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(\alpha) = y(\beta) = 0, \qquad \varepsilon \to 0^+$$

First we introduce a transformation $y = g(x)w$ to remove the y' term, which gives

$$2\varepsilon g' + a g = 0, \qquad g(x) = \exp\left(-\frac{1}{2\varepsilon}\int_\alpha^x a(s)\,\mathrm{d}s\right)$$

The ODE becomes

$$\varepsilon w'' + f w = 0, \qquad f(x) = b(x) - \frac{1}{2}a'(x) - \frac{1}{4\varepsilon}a^2(x)$$

The boundary values are still zero with $w(\alpha) = w(\beta) = 0$. Multiply by $w(x)$ and integrate the equation. Using the boundary conditions, we obtain

$$\varepsilon \int_\alpha^\beta [w'(x)]^2\,\mathrm{d}x = \int_\alpha^\beta f(x) w^2(x)\,\mathrm{d}x$$

If $f(x) \le 0$, then $w = 0$ is the unique trivial solution of BVP$(0,0)$. To satisfy this requirement, we notice two different cases.

$$|a(x)| \ge m > 0, \qquad f(x) \sim -\frac{1}{4\varepsilon}a^2(x) \le 0, \qquad \varepsilon \to 0^+$$
$$b(x) - \frac{1}{2}a'(x) \le 0, \qquad f(x) = b(x) - \frac{1}{2}a'(x) - \frac{1}{4\varepsilon}a^2(x) \le 0$$

- Boundary layers (8.2)

We start with the following example

$$\varepsilon y'' + (1-\varepsilon)y' - (1-\varepsilon)y = 0, \qquad y(0) = y(1) = 1$$

The characteristic roots are

$$m_\pm(\varepsilon) = \frac{\varepsilon - 1 \pm \sqrt{1 + 2\varepsilon - 3\varepsilon^2}}{2\varepsilon}$$

As $\varepsilon \to 0^+$, only one of the two roots remains finite

$$m_+(\varepsilon) = 1 - \varepsilon + \varepsilon^2 - 2\varepsilon^3 + O(\varepsilon^4), \qquad m_-(\varepsilon) = -\frac{1}{\varepsilon} + \varepsilon - \varepsilon^2 + 2\varepsilon^3 - O(\varepsilon^4)$$

The general solution is

$$y(x;\varepsilon) = C_+ e^{m_+(\varepsilon)x} + C_- e^{m_-(\varepsilon)x}$$

The coefficients are solved from the boundary conditions as

$$C_+ = \frac{1 - e^{m_-(\varepsilon)}}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}}, \qquad C_- = \frac{e^{m_+(\varepsilon)} - 1}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}}$$

Outer solution

When $0 < x \le 1$ and let $\varepsilon \to 0^+$, we can decompose the solution as

$$\begin{aligned} y(x;\varepsilon) &= e^{m_+(\varepsilon)(x-1)} + e^{m_-(\varepsilon)x}\frac{e^{m_+(\varepsilon)} - 1}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}} + e^{m_-(\varepsilon)}\frac{e^{m_+(\varepsilon)(x-1)} - e^{m_+(\varepsilon)}}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}} \\ &= e^{m_+(\varepsilon)(x-1)} + o(\varepsilon^p), \qquad \forall p \in \mathbb{N}, \qquad \varepsilon \to 0^+ \end{aligned}$$

If we consider the asymptotic series by $\{\varepsilon^n\}$, we only need to expand the first term in its Taylor series. The **outer expansion** is obtained as

$$y_{\text{out}}(x;\varepsilon) = e^{x-1} - \varepsilon(x-1)e^{x-1} + \frac{\varepsilon^2}{2}(x^2-1)e^{x-1} + O(\varepsilon^3), \qquad x > 0, \qquad \varepsilon \to 0^+$$

If we do not know the exact solution, we can still obtain the outer solution by considering the formal power series of $y(x;\varepsilon)$ as

$$y(x;\varepsilon) = \sum_{n\ge 0} y_n(x)\varepsilon^n, \qquad y_0(1) = 1, \qquad y_n(1) = 0, \qquad n \ge 1$$

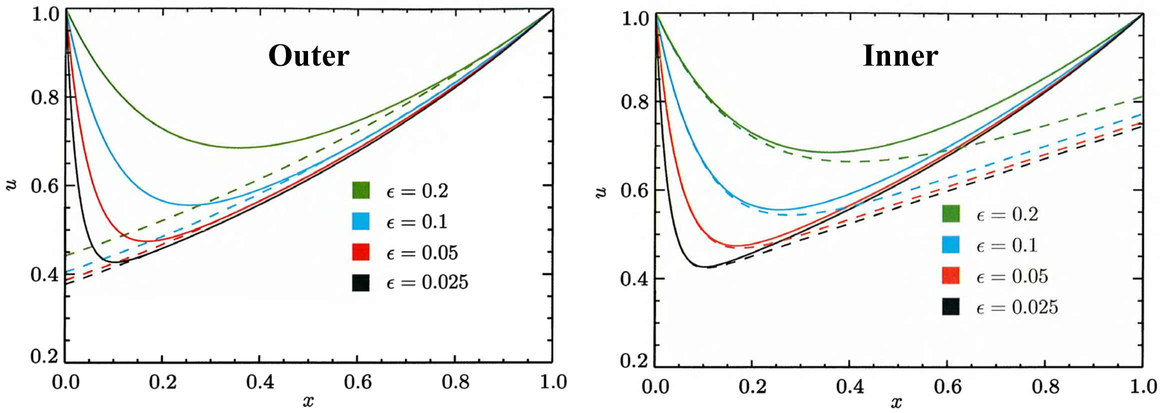
Note that we can only consider one of the two boundary conditions, since

$$\varepsilon^0:\ y_0' - y_0 = 0, \qquad y_0(1) = 1, \qquad y_0(x) = e^{x-1}$$

It is a first-order ODE. In general, we cannot obtain a solution that satisfies both boundary conditions for this type of problem. However, using the condition at $x = 1$ successfully gives the outer solution.

$$\varepsilon^1: \; y_0'' + y_1' - y_1 = 0, \qquad y_1(1) = 0, \qquad y_1(x) = -(x-1)e^{x-1}$$

Using the condition at $x = 0$ will not lead to a meaningful result. The comparison of the exact solution (solid) and the outer expansion (dashed) is shown below.



Inner solution

The outer expansion fails near $x = 0$ because the term $e^{-x/\varepsilon}$ that arises from $m_-(\varepsilon)$ does not converge uniformly for $x = O(\varepsilon)$ and for a fixed $0 < x \le 1$. To enlarge the thin boundary layer, we introduce an **inner variable** z, and the ODE becomes

$$z = \frac{x}{\varepsilon}, \qquad Y(z;\varepsilon) = y(\varepsilon z;\varepsilon), \qquad Y'' + (1-\varepsilon)Y' - \varepsilon(1-\varepsilon)Y = 0$$

For our example, given the exact solution, we have

$$Y(z;\varepsilon) = e^{m_+(\varepsilon)(\varepsilon z - 1)} + e^{m_-(\varepsilon)\varepsilon z}\frac{e^{m_+(\varepsilon)} - 1}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}} + e^{m_-(\varepsilon)}\frac{e^{m_+(\varepsilon)(\varepsilon z - 1)} - e^{m_+(\varepsilon)}}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}}$$

The term $e^{m_-(\varepsilon)\varepsilon z}$ is no longer singular in the exponent as $\varepsilon \to 0^+$. For a fixed $z \ge 0$, we have

$$Y(z;\varepsilon) = e^{m_+(\varepsilon)(\varepsilon z - 1)} + e^{m_-(\varepsilon)\varepsilon z}\left(1 - e^{-m_+(\varepsilon)}\right) + o(\varepsilon^p), \qquad \forall p \in \mathbb{N}$$

Again, consider the asymptotic series by $\{\varepsilon^n\}$, the **inner expansion** is

$$Y(z;\varepsilon) = \left[\left(1 - \frac{1}{e}\right)e^{-z} + \frac{1}{e}\right] + \frac{z + 1 - e^{-z}}{e}\varepsilon + O(\varepsilon^2)$$

$$y_{\text{in}}(x;\varepsilon) = \left[\left(1 - \frac{1}{e}\right)e^{-\frac{x}{\varepsilon}} + \frac{1}{e}\right] + \frac{\varepsilon}{e}\left(\frac{x}{\varepsilon} + 1 - e^{-\frac{x}{\varepsilon}}\right) + O(\varepsilon^2), \qquad \varepsilon \to 0^+$$

- Outer asymptotics (8.3)

We use the formal power series to study the general BVP(A, B) given as

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(x;\varepsilon) \sim S(x) = \sum_{n \ge 0} y_n(x)\varepsilon^n$$

For each order of ε, we have

$$\varepsilon^0:\ ay_0' + by_0 = 0, \qquad y_0 = C_0 \exp\left(-\int_\alpha^x \frac{b(s)}{a(s)}\,\mathrm{d}s\right)$$

$$\varepsilon^n:\ y_{n-1}'' + ay_n' + by_n = 0$$

Let D be the domain of convergence of $S(x)$. If $\alpha \in D$, the condition $y(\alpha) = A$ can be used to determine the coefficients $\{C_n\}$, with $y_0(\alpha) = A$ and $y_n(\alpha) = 0$ for $n \geq 1$. Hence we obtain an outer solution around α. Similarly, we can solve the case with $\beta \in D$.

Depending on the domain D, we have two simple scenarios. First, if $D = [\alpha, \beta)$ or $D = (\alpha, \beta]$, then a boundary layer is present at the endpoint. If $D = [\alpha, x_0) \cup (x_0, \beta]$, then an internal layer is present at the transition point x_0.

- Inner asymptotics for boundary and internal layers (8.4)

Denote the layer thickness around x_0 as $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$. The inner variable is

$$z = \frac{x - x_0}{\delta(\varepsilon)}, \qquad |z| \leq 1$$

With this rescaling, the ODE becomes

$$Y(z;\varepsilon) = y(x_0 + z\delta(\varepsilon);\varepsilon), \qquad \frac{\varepsilon}{\delta^2(\varepsilon)}Y'' + \frac{a}{\delta(\varepsilon)}Y' + bY = 0$$

Note that $a(z)$ and $b(z)$ are given as

$$a\big(x_0 + z\delta(\varepsilon)\big) = a(x_0) + a'(x_0)z\delta(\varepsilon) + O\big(\delta^2(\varepsilon)\big)$$

If $a(x_0) \neq 0$, the dominant balance gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{a(x_0)}{\delta(\varepsilon)}, \qquad \delta(\varepsilon) \sim \varepsilon$$

If $a(x_0) = 0$ but $a'(x_0) \neq 0$, the dominant balance gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim a'(x_0)z \sim b(x_0), \qquad \delta(\varepsilon) \sim \sqrt{\varepsilon}$$

The scaling relation of the layer thickness $\delta(\varepsilon)$ can be analyzed in general from the Taylor series of $a(z)$ and $b(z)$.

Assume $a(x_0) \neq 0$ for simplicity, we take $\delta(\varepsilon) = \varepsilon$ and the ODE becomes

$$Y'' + aY' + \varepsilon bY = 0, \qquad Y(z;\varepsilon) = \sum_{n\geq 0} Y_n(z)\varepsilon^n$$

The leading-order inner equation is

$$\varepsilon^0:\ Y_0'' + a(x_0)Y_0' = 0, \qquad Y_0(z) = c_1 + c_2 e^{-a(x_0)z}$$

In order to match with a reasonable outer solution, we need to choose the exponential decay solution within the inner layer. The sign of $a(x_0)$ then governs the existence of possible boundary or internal layers:

- Boundary layer at the left endpoint $x = \alpha$ with $z > 0$ can exist when $a(\alpha) > 0$
- Boundary layer at the right endpoint $x = \beta$ with $z < 0$ can exist when $a(\beta) < 0$
- Internal layer at an interior point $x = x_0$ can exists when $a(x_0) = 0$, but a different scaling may be required to achieve the dominant balance

➢ Matching of inner and outer asymptotic expansions (8.5)

Consider a possible boundary point $x_0 \in [\alpha, \beta]$ with thickness $\delta(\varepsilon)$. The inner solution $Y(z;\varepsilon)$ and the outer solution $y(x;\varepsilon)$ are given as

$$Y(z;\varepsilon) = \sum_{n\geq 0} Y_n(z)\mu_n(\varepsilon), \qquad z = \frac{x - x_0}{\delta(\varepsilon)}, \qquad y(x;\varepsilon) = \sum_{n\geq 0} y_n(x)\varepsilon^n$$

We introduce an intermediate variable w defined as

$$w = \frac{x - x_0}{\chi(\varepsilon)} = \frac{\delta(\varepsilon)}{\chi(\varepsilon)} z, \qquad \chi(\varepsilon) \to 0, \qquad \frac{\delta(\varepsilon)}{\chi(\varepsilon)} \to 0, \qquad \varepsilon \to 0^+$$

The **intermediate scale** $\chi(\varepsilon)$ is limited by $\delta(\varepsilon) \ll \chi(\varepsilon) \ll 1$, and it can define an overlap domain to connect the inner and outer expansions. Now we truncate both solutions as

$$Y_M(z;\varepsilon) = \sum_{m=0}^{M} Y_m(z)\mu_m(\varepsilon), \qquad y_N(x;\varepsilon) = \sum_{n=0}^{N} y_n(x)\varepsilon^n$$

We want to find a matched expansion $y_{\text{match}}^{NM}(w;\varepsilon)$ such that

$$y_N(x(w);\varepsilon) = y_{\text{match}}^{NM}(w;\varepsilon) + o\big(\mu_M(\varepsilon)\big), \qquad Y_M(z(w);\varepsilon) = y_{\text{match}}^{NM}(w;\varepsilon) + o(\varepsilon^N)$$

Specifically, if there is only one boundary point x_0, we can construct a single formula uniformly valid for the whole interval $[\alpha, \beta]$ as

$$y_{\text{unif}}^{NM}(x;\varepsilon) = y_N(x;\varepsilon) + Y_M\left(\frac{x - x_0}{\delta(\varepsilon)};\varepsilon\right) - y_{\text{match}}^{NM}\left(\frac{x - x_0}{\chi(\varepsilon)};\varepsilon\right)$$

➢ Examples (8.6)

Example 1. Matching of asymptotics

$$\varepsilon y'' + (1 + x^2)y' + xy = 0, \qquad y(-1) = 0, \qquad y(1) = 2$$

Since $a(x) = 1 + x^2 > 0$, we have the left endpoint $x_0 = -1$ as a boundary point. The outer expansion can be solved as

$$\varepsilon^0\colon\ (1 + x^2)y_0' + xy_0 = 0, \qquad y_0(1) = 2, \qquad y_0(x) = 2\sqrt{\frac{2}{1 + x^2}}$$

$$\varepsilon^n:\ y_{n-1}'' + (1+x^2)y_n' + xy_n = 0, \qquad y_n(1) = 0$$

The solution of $y_1(x)$ can be obtained as

$$y_1(x) = \frac{1}{16}\sqrt{\frac{2}{1+x^2}}\left[\frac{24x}{(1+x^2)^2} + \frac{4x}{1+x^2} + 4\arctan x - \pi - 8\right]$$

For the inner expansion, with $a(-1) = 2 \neq 0$, we take $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$z = \frac{x+1}{\varepsilon}, \qquad Y'' + (2 - 2\varepsilon z + \varepsilon^2 z^2)Y' + \varepsilon(\varepsilon z - 1)Y = 0$$

With the choice $\mu_n(\varepsilon) = \varepsilon^n$, the inner expansion can be solved as

$$\varepsilon^0:\ Y_0'' + 2Y_0' = 0, \qquad Y_0(0) = 0, \qquad Y_0(z) = c_1(1 - e^{-2z})$$

$$\varepsilon^1:\ Y_1'' + 2Y_1' - 2zY_0' - Y_0 = 0, \qquad Y_1(0) = 0$$

The solution of $Y_1(z)$ can be obtained as

$$Y_1(z) = \frac{c_1}{4}(2z-1) + d_1 - \left[\frac{c_1}{4}(4z^2 + 2z + 1) + d_2\right]e^{-2z}, \qquad d_1 - d_2 = \frac{c_1}{2}$$

Now we need to find the intermediate scale $\chi(\varepsilon)$ to match the two expansions.

$$w = \frac{x+1}{\chi(\varepsilon)} = \frac{\varepsilon z}{\chi(\varepsilon)}, \qquad \varepsilon \ll \chi(\varepsilon) \ll 1$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. The outer expansion becomes

$$y_0(w) = 2\sqrt{\frac{2}{1+(\chi w - 1)^2}} = 2 + \chi(\varepsilon)w + O(\chi(\varepsilon)^2), \qquad y_1(w) = -1 - \frac{\pi}{8} + O\big(\chi(\varepsilon)\big)$$

$$y_{\text{out}}(x(w);\varepsilon) = 2 + \chi(\varepsilon)w - \frac{8+\pi}{8}\varepsilon + O\big(\varepsilon\chi(\varepsilon)\big) + O(\chi(\varepsilon)^2)$$

The inner expansion becomes

$$Y_0(w) = c_1\left(1 - e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right) = c_1 + O\left(e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_1(w) = \frac{c_1 w}{2}\frac{\chi(\varepsilon)}{\varepsilon} + d_1 - \frac{c_1}{4} + O\left(\frac{\chi^2(\varepsilon)}{\varepsilon^2}e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

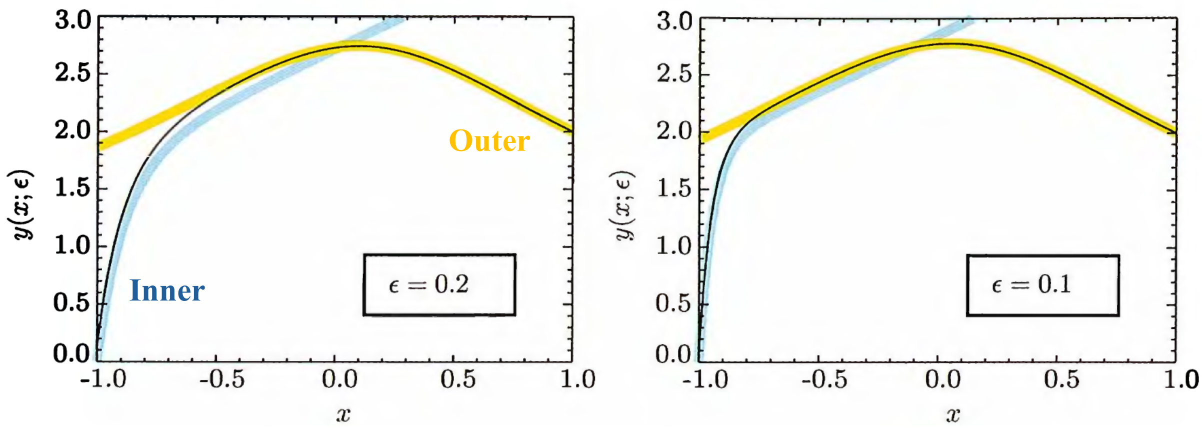
$$Y_{\text{in}}(z(w);\varepsilon) = c_1 + \frac{c_1 w}{2}\chi(\varepsilon) + \left(d_1 - \frac{c_1}{4}\right)\varepsilon + O\left(e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right) + O\left(\frac{\chi^2(\varepsilon)}{\varepsilon}e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

We need to properly choose $\chi(\varepsilon)$ so that all $O(\cdot)$ terms can be controlled. Note that they first should be $o(\varepsilon)$, which gives

$$\chi^2(\varepsilon) \ll \varepsilon, \qquad e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}} \ll \varepsilon, \qquad \varepsilon \ln \varepsilon^{-1} \ll \chi(\varepsilon) \ll \sqrt{\varepsilon}$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$c_1 = 2, \qquad d_1 = -\left(\frac{1}{2} + \frac{\pi}{8}\right)$$



Therefore, the approximation in the overlap domain is

$$y_{\text{match}}^{1,1}(x;\varepsilon) = 2 + \chi(\varepsilon)w - \frac{8+\pi}{8}\varepsilon = x + 3 - \frac{8+\pi}{8}\varepsilon$$

The uniformly valid approximation is

$$y_{\text{unif}}^{1,1}(x;\varepsilon) = y_{\text{out}}(x;\varepsilon) + Y_{\text{in}}\left(\frac{x+1}{\varepsilon};\varepsilon\right) - y_{\text{match}}^{1,1}(x;\varepsilon)$$

This result does not satisfy the ODE and boundary conditions, but the error is very small.

Example 2. Different scaling of layer thickness

$$\varepsilon y'' + 12x^{1/3}y' + y = 0, \qquad y(0) = y(1) = 1$$

In this case, we have

$$b(x) - \frac{1}{2}a'(x) = 1 - 2x^{-\frac{2}{3}} \leq -1$$

This shows that the BVP has a unique solution. Since $a(x) > 0$ for $x \in (0,1]$, there will be no possible boundary point in this interval, hence the only possible boundary point is $x_0 = 0$. The outer expansion can be solved as

$$\varepsilon^0\colon\ 12x^{1/3}y_0' + y_0 = 0, \qquad y_0(1) = 1, \qquad y_0(x) = \exp\left(\frac{1-x^{2/3}}{8}\right)$$

For the inner expansion, we need to use $\delta(\varepsilon)$ and find its proper scaling.

$$z = \frac{x}{\delta(\varepsilon)}, \qquad \frac{\varepsilon}{\delta^2(\varepsilon)}Y'' + \frac{12\big(z\delta(\varepsilon)\big)^{1/3}}{\delta(\varepsilon)}Y' + Y = 0$$

The dominant balance gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim [\delta(\varepsilon)]^{-2/3}, \qquad \delta(\varepsilon) \sim \varepsilon^{3/4}$$

The inner equation then becomes

$$Y'' + 12z^{\frac{1}{3}}Y' + \sqrt{\varepsilon}Y = 0$$

$$\varepsilon^0\colon\ Y_0'' + 12z^{1/3}Y_0' = 0, \qquad Y_0(0) = 1$$

The solution can be obtained as

$$Y_0(z) = 1 + C\int_0^z e^{-9s^{4/3}}\mathrm{d}s$$

The intermediate scale can be denoted as $\chi(\varepsilon) = \varepsilon^p$, and we have

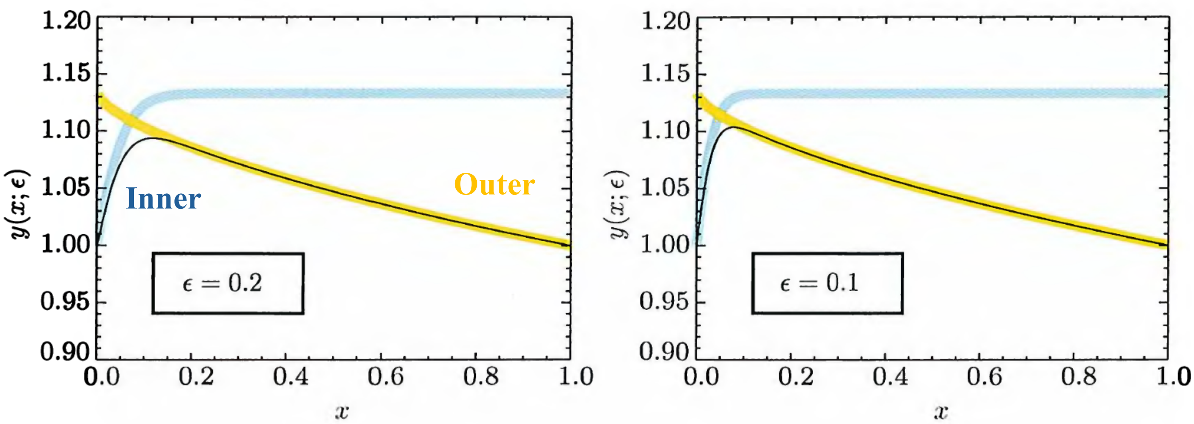
$$w = \frac{x}{\varepsilon^p} = \frac{\varepsilon^{3/4}z}{\varepsilon^p}, \qquad \varepsilon^{3/4} \ll \varepsilon^p \ll 1, \qquad 0 < p < \frac{3}{4}$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. Since we keep the leading-order term, the matching condition can be simply written as

$$\lim_{x\to 0^+} y_0(x) = \lim_{z\to +\infty} Y_0(z)\,, \qquad e^{1/8} = 1 + \frac{C}{4\sqrt{3}}\Gamma\left(\frac{3}{4}\right)$$

The uniformly valid approximation is

$$y_{\mathrm{unif}}^{0,0}(x;\varepsilon) = \exp\left(\frac{1-x^{2/3}}{8}\right) + 1 + C\int_0^{x\varepsilon^{-3/4}} e^{-9s^{4/3}}\mathrm{d}s - e^{1/8}$$



Example 3. Internal layer

$$\varepsilon y'' + xy' - \left(1+\frac{x}{4}\right)y = 0, \qquad y(-1) = 3, \qquad y(1) = 1$$

In this case, we have

$$b(x) - \frac{1}{2}a'(x) = -\frac{3}{2} - \frac{x}{4} \le -\frac{5}{4}$$

The only possible boundary point is $x_0 = 0$ where $a(0) = 0$. This corresponds to an internal layer point. As $a'(0) \neq 0$, the layer thickness scales as $\delta(\varepsilon) \sim \sqrt{\varepsilon}$. The outer expansions need to be solved for both regions to the left and right of $x_0 = 0$.

$$\varepsilon^0:\ xy_0' - \left(1+\frac{x}{4}\right)y_0 = 0, \qquad y_L(-1) = 3, \qquad y_R(1) = 1$$

$$y_{L0}(x) = -3xe^{\frac{x+1}{4}}, \qquad y_{R0}(x) = xe^{\frac{x-1}{4}}$$

For the inner expansion, we take $\delta(\varepsilon) = \sqrt{\varepsilon}$, $\mu_n(\varepsilon) = \sqrt{\varepsilon}$ and the inner equation is

$$z = \frac{x}{\sqrt{\varepsilon}}, \qquad Y'' + zY' - \left(1 + \frac{z\sqrt{\varepsilon}}{4}\right)Y = 0$$

$$\varepsilon^0\colon\ Y_0'' + zY_0' - Y_0 = 0, \qquad Y_0(z) = C_1 z + C_2\left(e^{-\frac{z^2}{2}} + z\int_{-\infty}^{z} e^{-\frac{s^2}{2}} \mathrm{d}s\right)$$

The two coefficients are to be determined from the matching conditions. The intermediate scale can be chosen as $\chi(\varepsilon) = \varepsilon^{1/4}$, and for a fixed w we have

$$w = \frac{x}{\varepsilon^{1/4}} = \frac{\sqrt{\varepsilon}z}{\varepsilon^{1/4}}, \qquad x \to 0, \qquad z \to \operatorname{sgn}(w)\cdot\infty, \qquad \varepsilon \to 0^+$$

The inner and outer expansions become

$$y_{L0}(w) = -3\varepsilon^{1/4} w e^{1/4} + O(\sqrt{\varepsilon}), \qquad y_{R0}(w) = \varepsilon^{1/4} w e^{-1/4} + O(\sqrt{\varepsilon})$$

$$Y_0(w^-) = C_1 \varepsilon^{-1/4} w + O\left(e^{-w^2/\sqrt{\varepsilon}}\right), \qquad Y_0(w^+) = \left(C_1 + C_2\sqrt{2\pi}\right)\varepsilon^{-1/4} w + O\left(e^{-w^2/\sqrt{\varepsilon}}\right)$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$C_1 = -3e^{\frac{1}{4}}\sqrt{\varepsilon}, \qquad C_2 = \frac{e^{-1/4} + 3e^{1/4}}{\sqrt{2\pi}}\sqrt{\varepsilon}$$

The coefficients depend on ε. The approximations in the overlap domain are

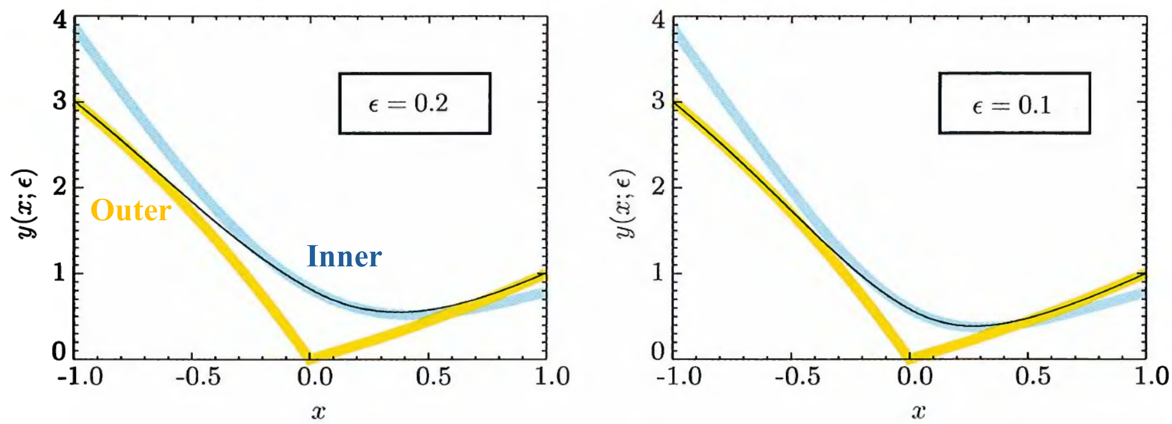
$$y_{L,\text{match}}^{0,0}(x;\varepsilon) = -3e^{1/4}x, \qquad y_{R,\text{match}}^{0,0}(x;\varepsilon) = e^{-1/4}x$$

The uniformly valid approximation is

$$y_{\text{unif}}^{0,0}(x;\varepsilon) = y_{L0}(x) + Y_0\left(\frac{x}{\sqrt{\varepsilon}}\right) - y_{L,\text{match}}^{0,0}(x;\varepsilon), \qquad x < 0$$

$$y_{\text{unif}}^{0,0}(x;\varepsilon) = y_{R0}(x) + Y_0\left(\frac{x}{\sqrt{\varepsilon}}\right) - y_{R,\text{match}}^{0,0}(x;\varepsilon), \qquad x > 0$$

An internal layer like this is also called a corner layer.



- Exercise

Right boundary layer

$$\varepsilon y'' - y' + x^4 y = 0, \qquad y(-1) = y(1) = 1$$

Since $a(x) = -1 < 0$, we have a right boundary point at $x = 1$. The outer expansion is

$$\varepsilon^0: -y_0' + x^4 y_0 = 0, \qquad y_0(-1) = 1, \qquad y_0(x) = e^{\frac{x^5+1}{5}}$$

$$\varepsilon^1: y_0'' - y_1' + x^4 y_1 = 0, \qquad y_1(-1) = 0$$

The solution of $y_1(x)$ can be obtained as-

$$y_1(x) = \frac{1}{9} e^{\frac{x^5+1}{5}} (-8 + 9x^4 + x^9)$$

For the inner expansion, with $a(1) \neq 0$, we take $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$z = \frac{x-1}{\varepsilon} < 0, \qquad Y'' - Y' + \varepsilon(\varepsilon z + 1)^4 Y = 0$$

With the choice $\mu_n(\varepsilon) = \varepsilon^n$, the inner expansion can be solved as

$$\varepsilon^0: Y_0'' - Y_0' = 0, \qquad Y_0(0) = 1, \qquad Y_0(z) = 1 + c_1(e^z - 1)$$

$$\varepsilon^1: Y_1'' - Y_1' + Y_0 = 0, \qquad Y_1(0) = 0$$

The solution of $Y_1(z)$ can be obtained as

$$Y_1(z) = c_1(e^z - z - 1 - ze^z) + z + c_2(e^z - 1)$$

Now we need to find the intermediate scale $\chi(\varepsilon)$ to match the two expansions.

$$w = \frac{x-1}{\chi(\varepsilon)} = \frac{\varepsilon z}{\chi(\varepsilon)}, \qquad \varepsilon \ll \chi(\varepsilon) \ll 1$$

We assume a fixed $w < 0$ and let $\varepsilon \to 0^+$. The outer expansion becomes

$$y_0(w) = e^{2/5} + e^{2/5}\chi(\varepsilon) w + O(\chi(\varepsilon)^2), \qquad y_1(w) = \frac{2}{9} e^{2/5} + O\big(\chi(\varepsilon)\big)$$

$$y_{\text{out}}(x(w); \varepsilon) = e^{2/5} + e^{2/5}\chi(\varepsilon) w + \frac{2}{9} e^{2/5}\varepsilon + O\big(\varepsilon\chi(\varepsilon)\big) + O(\chi(\varepsilon)^2)$$

The inner expansion becomes

$$Y_0(w) = 1 - c_1 + O\left(e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_1(w) = (1 - c_1) w \frac{\chi(\varepsilon)}{\varepsilon} - (c_1 + c_2) + O\left(\frac{\chi(\varepsilon)}{\varepsilon} e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_{\text{in}}(z(w); \varepsilon) = 1 - c_1 + (1 - c_1) w\chi(\varepsilon) - (c_1 + c_2)\varepsilon + O\left(e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right) + O\left(\chi(\varepsilon) e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

We need to properly choose $\chi(\varepsilon)$ so that all $O(\cdot)$ terms can be controlled. Note that they first should be $o(\varepsilon)$, which gives

$$\chi^2(\varepsilon) \ll \varepsilon, \qquad e^{w\frac{\chi(\varepsilon)}{\varepsilon}} \ll \varepsilon, \qquad \varepsilon \ln \varepsilon^{-1} \ll \chi(\varepsilon) \ll \sqrt{\varepsilon}$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$c_1 = 1 - e^{2/5}, \qquad c_2 = \frac{7}{9}e^{2/5} - 1$$

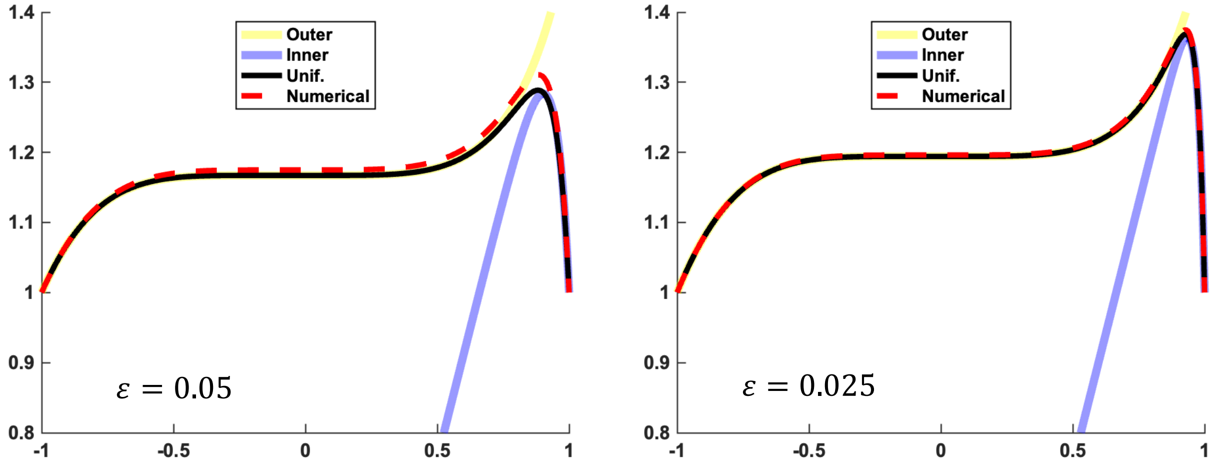
Therefore, the approximation in the overlap domain is

$$y^{1,1}_{\text{match}}(x;\varepsilon) = xe^{2/5} + \frac{2}{9}e^{2/5}\varepsilon$$

The uniformly valid approximation is

$$y^{1,1}_{\text{unif}}(x;\varepsilon) = y_{\text{out}}(x;\varepsilon) + Y_{\text{in}}\left(\frac{x-1}{\varepsilon};\varepsilon\right) - y^{1,1}_{\text{match}}(x;\varepsilon)$$

As ε becomes smaller, the approximation is closer to the numerically solved result.



Left boundary layer

$$\varepsilon y'' + y' - xy = 0, \qquad y(0) = 0, \qquad y(1) = e^{1/2}$$

Since $a(x) = 1 > 0$, we have a left boundary point at $x = 0$. The outer expansion is

$$\varepsilon^0\text{: } y_0' - xy_0 = 0, \qquad y_0(1) = e^{\frac{1}{2}}, \qquad y_0(x) = e^{\frac{x^2}{2}}$$

$$\varepsilon^1\text{: } y_0'' + y_1' - xy_1 = 0, \qquad y_1(1) = 0, \qquad y_1(x) = -\frac{1}{3}e^{\frac{x^2}{2}}(-4 + 3x + x^3)$$

For the inner expansion, with $a(1) \neq 0$, we take $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$z = \frac{x}{\varepsilon} > 0, \qquad Y'' + Y' - \varepsilon(\varepsilon z + 1)Y = 0$$

With the choice $\mu_n(\varepsilon) = \varepsilon^n$, the inner expansion can be solved as

$$\varepsilon^0\text{: } Y_0'' + Y_0' = 0, \qquad Y_0(0) = 0, \qquad Y_0(z) = c_1(1 - e^{-z})$$

$$\varepsilon^1\text{: } Y_1'' + Y_1' - Y_0 = 0, \qquad Y_1(0) = 0$$

The solution of $Y_1(z)$ can be obtained as

$$Y_1(z) = c_1(e^{-z} + z - 1 + ze^{-z}) + z + c_2(1 - e^{-z})$$

Now we need to find the intermediate scale $\chi(\varepsilon)$ to match the two expansions.

$$w = \frac{x}{\chi(\varepsilon)} = \frac{\varepsilon z}{\chi(\varepsilon)}, \qquad \varepsilon \ll \chi(\varepsilon) \ll 1$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. The outer expansion becomes

$$y_0(w) = 1 + O(\chi(\varepsilon)^2), \qquad y_1(w) = \frac{4}{3} - \chi(\varepsilon)w + O\big(\chi^2(\varepsilon)\big)$$

$$y_{\text{out}}(x(w);\varepsilon) = 1 + \frac{4}{3}\varepsilon - \chi(\varepsilon)w\varepsilon + O\big(\varepsilon\chi^2(\varepsilon)\big) + O(\chi(\varepsilon)^2)$$

The inner expansion becomes

$$Y_0(w) = c_1 + O\left(e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_1(w) = (1 + c_1)w\frac{\chi(\varepsilon)}{\varepsilon} + (c_2 - c_1) + O\left(\frac{\chi(\varepsilon)}{\varepsilon}e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_{\text{in}}(z(w);\varepsilon) = c_1 + (1 + c_1)w\chi(\varepsilon) + (c_2 - c_1)\varepsilon + O\left(e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right) + O\left(\chi(\varepsilon)e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

We can similarly choose $\chi(\varepsilon)$ so that all $O(\cdot)$ terms are $o(\varepsilon)$. Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$c_1 = -1 - \varepsilon, \qquad c_2 = \frac{2}{\varepsilon} - \varepsilon + \frac{4}{3}$$

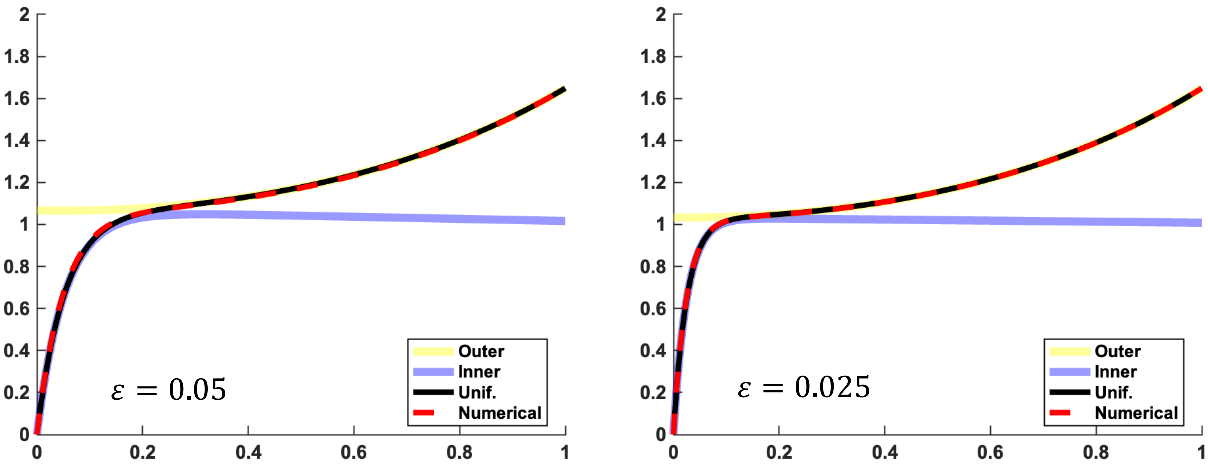
Therefore, the approximation in the overlap domain is

$$y^{1,1}_{\text{match}}(x;\varepsilon) = 1 + \frac{4}{3}\varepsilon - \varepsilon x$$

The uniformly valid approximation is

$$y^{1,1}_{\text{unif}}(x;\varepsilon) = y_{\text{out}}(x;\varepsilon) + Y_{\text{in}}\left(\frac{x}{\varepsilon};\varepsilon\right) - y^{1,1}_{\text{match}}(x;\varepsilon)$$

As ε becomes smaller, the approximation is closer to the numerically solved result.



For this problem we can obtain an exact solution. To facilitate comparison, we first write down

$$y_0(x) = e^{x^2/2}, \qquad Y_0(z) = 1 - e^{-z}, \qquad y^{0,0}_{\text{unif}}(x;\varepsilon) = e^{x^2/2} - e^{-x/\varepsilon}$$

We first introduce the transform to remove the y' term as

$$\varepsilon\xi'' - x\xi - \frac{\xi}{4\varepsilon} = 0, \qquad y(x) = \xi(x)\exp\left(-\frac{x}{2\varepsilon}\right)$$

Then with a new independent variable, the ODE becomes

$$t = \varepsilon^{-\frac{1}{3}}\left(x + \frac{1}{4\varepsilon}\right), \qquad \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - t\xi = 0$$

This is the Airy equation. The boundary conditions are modified to

$$\xi(t_0) = \xi\left(\frac{1}{4}\varepsilon^{-\frac{4}{3}}\right) = 0, \qquad \xi(t_1) = \xi\left(\varepsilon^{-\frac{1}{3}} + \frac{1}{4}\varepsilon^{-\frac{4}{3}}\right) = \exp\left(\frac{1+\varepsilon}{2\varepsilon}\right)$$

We can obtain the exact solution of $y(x)$ as follows

$$y(x) = \exp\left(\frac{\varepsilon + 1 - x}{2\varepsilon}\right) \cdot \frac{\mathrm{Ai}(t_0)\mathrm{Bi}\big(t(x)\big) - \mathrm{Ai}\big(t(x)\big)\mathrm{Bi}(t_0)}{\mathrm{Ai}(t_0)\mathrm{Bi}(t_1) - \mathrm{Ai}(t_1)\mathrm{Bi}(t_0)}$$

The asymptotic expansions for Airy functions are

$$\mathrm{Ai}(t) \sim \frac{t^{-1/4}}{2\sqrt{\pi}}e^{-\zeta}, \qquad \mathrm{Bi}(t) \sim \frac{t^{-1/4}}{\sqrt{\pi}}e^{\zeta}, \qquad \zeta(t) = \frac{2}{3}t^{3/2}, \qquad t \to +\infty$$

For the outer expansion, assumed a fixed x and let $\varepsilon \to 0^+$. Denote $\delta = t(x) - t_0$ and we have

$$\zeta\big(t(x)\big) - \zeta(t_0) = t_0^{1/2}\delta + \frac{1}{4}t_0^{-1/2}\delta^2 + O\big(t_0^{-3/2}\delta^3\big) = \frac{x}{2\varepsilon} + \frac{x^2}{2} + O(\varepsilon x^3)$$

$$\zeta(t_1) - \zeta(t_0) = \zeta\big(t(1)\big) - \zeta(t_0) = \frac{1}{2\varepsilon} + \frac{1}{2} + O(\varepsilon)$$

Therefore, we have

$$y(x) \sim \exp\left(\frac{\varepsilon + 1 - x}{2\varepsilon}\right) \cdot \left(\frac{t(x)}{t_1}\right)^{\frac{1}{4}} \cdot \frac{\sinh\big[\zeta\big(t(x)\big) - \zeta(t_0)\big]}{\sinh[\zeta(t_1) - \zeta(t_0)]}$$

$$\sim \exp\left(\frac{\varepsilon + 1 - x}{2\varepsilon}\right) \cdot \exp\left(\frac{x}{2\varepsilon} + \frac{x^2}{2} - \frac{1}{2\varepsilon} - \frac{1}{2}\right) \sim e^{\frac{x^2}{2}}, \qquad \varepsilon \to 0^+$$

For the inner expansion, assumed as fixed $z = x/\varepsilon$ and let $\varepsilon \to 0^+$. Now we have

$$\zeta\big(t(z)\big) - \zeta(t_0) = t_0^{1/2}\delta + \frac{1}{4}t_0^{-1/2}\delta^2 + O\big(t_0^{-3/2}\delta^3\big) = \frac{z}{2} + \frac{\varepsilon^2 z^2}{2} + O(\varepsilon^4 z^3)$$

The asymptotic behavior becomes

$$y(z) \sim \exp\left(\frac{\varepsilon + 1 - \varepsilon z}{2\varepsilon}\right) \cdot \left(\frac{t(z)}{t_1}\right)^{\frac{1}{4}} \cdot \frac{\sinh\big[\zeta\big(t(z)\big) - \zeta(t_0)\big]}{\sinh[\zeta(t_1) - \zeta(t_0)]}$$

$$\sim 2e^{-\frac{z}{2}}\sinh\left(\frac{z}{2}\right) = 1 - e^{-z}, \qquad \varepsilon \to 0^+$$

The asymptotic behavior of the exact solution is consistent with the outer and inner expansions.

Different scaling of layer thickness

$$\varepsilon y'' + x^{3/2} y' - y = 0, \qquad y(0) = \alpha, \qquad y(1) = e^{2/5}$$

The BVP has a unique solution since $b(x) - a'(x)/2 < 0$. Also, with $a(x) > 0$ for $x \in (0,1]$, there will be no possible boundary point in this interval, hence the only possible one is $x_0 = 0$. The outer expansion can be solved as

$$\varepsilon^0\colon\ x^{3/2} y_0' - y_0 = 0, \qquad y_0(1) = e^{2/5}, \qquad y_0(x) = \exp\left(\frac{12}{5} - \frac{2}{\sqrt{x}}\right)$$

For the inner expansion, we need to use $\delta(\varepsilon)$ and find its proper scaling.

$$z = \frac{x}{\delta(\varepsilon)}, \qquad \frac{\varepsilon}{\delta^2(\varepsilon)} Y'' + \frac{\big(z\delta(\varepsilon)\big)^{3/2}}{\delta(\varepsilon)} Y' - Y = 0$$

We should take the dominant balance between the first and third term, which gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim 1, \qquad \delta(\varepsilon) \sim \sqrt{\varepsilon}$$

The inner equation then becomes

$$Y'' + z^{3/2}\varepsilon^{1/4} Y' - Y = 0, \qquad \varepsilon^0\colon\ Y_0'' - Y_0 = 0, \qquad Y_0(0) = \alpha$$

The solution can be obtained as

$$Y_0(z) = (\alpha - C)e^{z} + Ce^{-z}$$

The intermediate scale can be denoted as $\chi(\varepsilon) = \varepsilon^p$, and we have

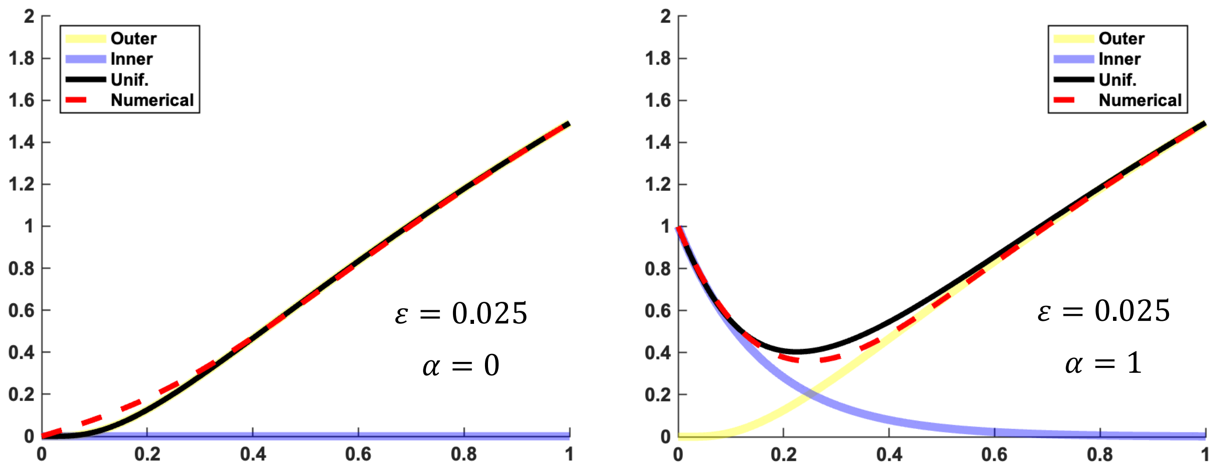
$$w = \frac{x}{\varepsilon^p} = \frac{\sqrt{\varepsilon} z}{\varepsilon^p}, \qquad \varepsilon^{1/2} \ll \varepsilon^p \ll 1, \qquad 0 < p < \frac{1}{2}$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. Since we keep the leading-order term, the matching condition can be simply written as

$$\lim_{x\to 0^+} y_0(x) = \lim_{z\to+\infty} Y_0(z)\,, \qquad 0 = \lim_{z\to+\infty} (\alpha - C)e^{z}\,, \qquad C = \alpha$$

The uniformly valid approximation is

$$y_{\text{unif}}^{0,0}(x;\varepsilon) = \exp\left(\frac{12}{5} - \frac{2}{\sqrt{x}}\right) + \alpha e^{-z}$$



Internal layer

$$\varepsilon y'' + 2xy' + \frac{\sin x}{2} y = 0, \qquad y(-1) = 1, \qquad y(1) = 2$$

This problem has a unique solution because

$$b(x) - \frac{1}{2}a'(x) = \frac{\sin x}{2} - 1 < 0, \qquad x \in [-1,1]$$

The only possible boundary point is $x_0 = 0$ where $a(0) = 0$. This corresponds to an internal layer point. As $a'(0) \neq 0$, the layer thickness scales as $\delta(\varepsilon) \sim \sqrt{\varepsilon}$. The outer expansions are

$$\varepsilon^0\text{: } 2xy_0' + \frac{\sin x}{2} y_0 = 0, \qquad y_L(-1) = 1, \qquad y_R(1) = 2$$

$$y_{L0}(x) = e^{-\frac{1}{4}[\mathrm{Si}(x)+\mathrm{Si}(1)]}, \qquad y_{R0}(x) = 2e^{-\frac{1}{4}[\mathrm{Si}(x)-\mathrm{Si}(1)]}, \qquad \mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\mathrm{d}t$$

For the inner expansion, we take $\delta(\varepsilon) = \sqrt{\varepsilon}$, $\mu_n(\varepsilon) = \sqrt{\varepsilon}$ and the inner equation is

$$z = \frac{x}{\sqrt{\varepsilon}}, \qquad Y'' + 2zY' + \frac{1}{2}\sin\left(z\sqrt{\varepsilon}\right) Y = 0$$

$$\varepsilon^0\text{: } Y_0'' + 2zY_0' = 0, \qquad Y_0(z) = C_1 + \frac{C_2}{\sqrt{\pi}}\int_{-\infty}^{z} e^{-s^2}\mathrm{d}s$$

The intermediate scale can be chosen as $\chi(\varepsilon) = \varepsilon^{1/4}$, and for a fixed w we have

$$w = \frac{x}{\varepsilon^{1/4}} = \frac{\sqrt{\varepsilon}z}{\varepsilon^{1/4}}, \qquad x \to 0, \qquad z \to \mathrm{sgn}(w)\cdot\infty, \qquad \varepsilon \to 0^+$$

The inner and outer expansions become

$$y_{L0}(w) = e^{-\mathrm{Si}(1)/4} + O\left(\varepsilon^{1/4}\right), \qquad y_{R0}(w) = 2e^{\mathrm{Si}(1)/4} + O\left(\varepsilon^{1/4}\right)$$

$$Y_0(w^-) = C_1 + O\left(e^{-w^2/\sqrt{\varepsilon}}\right), \qquad Y_0(w^+) = (C_1 + C_2) + O\left(e^{-w^2/\sqrt{\varepsilon}}\right)$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$C_1 = e^{-\frac{\mathrm{Si}(1)}{4}}, \qquad C_2 = 2e^{\frac{\mathrm{Si}(1)}{4}} - e^{-\frac{\mathrm{Si}(1)}{4}}$$

This internal layer corresponds to a rapid change in $y(x)$.