Fundamentals of Asymptotic Analysis

➢ Big-O and small-o notation (Book chapter 1.1)

For a domain Ω, consider general complex-valued functions $f, g: \Omega \to \mathbb{C}$.

$f(z) = O\big(g(z)\big), \quad z \in \Omega$	$\exists K > 0,\ \forall z \in \Omega,\ \lvert f(z)\rvert \le K\lvert g(z)\rvert$
$f(z) = O\big(g(z)\big), \quad z \to z_0$	$\exists \delta > 0,\ \forall z \in B(z_0, \delta),\ f(z) = O\big(g(z)\big)$
$f(z) = O\big(g(z)\big), \quad z \to \infty$	$\exists M > 0,\ \forall \lvert z\rvert > M,\ f(z) = O\big(g(z)\big)$
$f(z) = o\big(g(z)\big), \quad z \to z_0$	$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall z \in B(z_0, \delta),\ \lvert f(z)\rvert \le \varepsilon\lvert g(z)\rvert$
$f(z) = o\big(g(z)\big), \quad z \to \infty$	$\forall \varepsilon > 0,\ \exists M > 0,\ \forall \lvert z\rvert > M,\ \lvert f(z)\rvert \le \varepsilon\lvert g(z)\rvert$

The are several properties

- If $g(z) \neq 0$, then

$$f(z) = O\big(g(z)\big), \quad z \to z_0 \quad \Leftrightarrow \quad \limsup_{z \to z_0} \left|\frac{f(z)}{g(z)}\right| \text{ exists}$$

$$f(z) = o\big(g(z)\big), \quad z \to z_0 \quad \Leftrightarrow \quad \lim_{z \to z_0} \left|\frac{f(z)}{g(z)}\right| = 0$$

- If $f = o(g)$, then we have $f = O(g)$
- If $f = O(g), g = O(h)$, then $f = O(h)$. Similarly, if $f = o(g), g = o(h)$, then $f = o(h)$
- If $f_n = O(g_n)$ for $z \to z_0$ and $n = 1,2,\cdots,N$, then

$$\sum_{n=1}^{N} a_n f_n = O\left(\sum_{n=1}^{N} |a_n||g_n|\right), \qquad z \to z_0$$

- As $z \to z_0$, if $f_1 = O(g_1)$, $f_2 = O(g_2)$ and $g_1 = O(g_2)$, then $f_1 + f_2 = O(g_2)$
- If $g(x) \ge 0$, then

$$f(x) = O\big(g(x)\big) \quad \Rightarrow \quad \left|\int_a^y f(x)\,\mathrm{d}x\right| = O\left(\left|\int_a^y g(x)\,\mathrm{d}x\right|\right)$$

Note: The relation between derivatives cannot be guaranteed. Same for big-O notation.

$$f(x) = o\big(g(x)\big) \quad \not\Rightarrow \quad f'(x) = o\big(g'(x)\big), \qquad \text{e.g. } f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$$

The relation between antiderivatives requires $g \ge 0$. As a counterexample, consider $f(x) = 1$ and $g(x) = e^{ix}$. We have $f(x) \le g(x)$, but

$$\int_0^y f(x)\,\mathrm{d}x = y \to \infty, \qquad \left|\int_0^y g(x)\,\mathrm{d}x\right| = \left|e^{iy} - 1\right| \le 2, \qquad \text{as } y \to \infty$$

With these notations, the absolute and relative errors can be denoted as

$$f(z) = \tilde{f}(z) + o(1), \qquad z \to z_0$$

$$f(z) = \tilde{f}(z)\big(1 + o(1)\big), \qquad z \to z_0$$

➢ Asymptotic sequence and series (1.3)

Asymptotic sequence

If there is a sequence of functions $\{\phi_n : \Omega \to \mathbb{C}\}$ such that for any $n > m$, we have

$$\phi_n(z) = o\big(\phi_m(z)\big), \qquad z \to z_0$$

Then we call $\{\phi_n\}$ as an asymptotic sequence around z_0.

Example

$$\phi_n(z) = z^{n_0+n}, \qquad n_0 \in \mathbb{Z}, \qquad z = B(0, \delta)$$

Asymptotic series

For a function $f: \Omega \to \mathbb{C}$ and an asymptotic sequence $\{\phi_n\}$ around z_0, if there exists a sequence of constants $\{a_n\}$ such that for any $N \in \mathbb{N}$, we have

$$f(z) = \sum_{n=0}^{N} a_n \phi_n(z) + o\big(\phi_N(z)\big), \qquad z \to z_0$$

Then the formal infinite series below is an asymptotic series, and is an asymptotic expansion of $f(z)$ around $z = z_0$.

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \qquad z \to z_0$$

Note that:

- ♦ As a function series it is usually not convergent, and we need to truncate the sum.
- ♦ For a given $f(z)$, there can be many asymptotic series around z_0.
- ♦ For a given $\{\phi_n\}$ and $\{a_n\}$, the asymptotic series can correspond to different functions $f(z)$ and $g(z)$ as long as $f - g = o(\phi_n)$. As an example, we can choose

$$f - g = e^{-\frac{1}{x^2}} = o(x^n), \qquad x \to 0$$

- ♦ For a given f and $\{\phi_n\}$, the sequence $\{a_n\}$ can be uniquely determined as

$$a_N = \lim_{z \to z_0} \frac{f(z) - \sum_{n=0}^{N-1} a_n \phi_n(z)}{\phi_N(z)}$$

- ♦ Not every f can be expanded using $\{\phi_n\}$

Existence theorem (1.4)

For an asymptotic series $\Phi = \{\phi_n\}$, denote

$$\mathcal{F}_\Phi = \left\{ f: \Omega \to \mathbb{C} \mid \exists \{a_n\} \text{ s.t. } f(z) \sim \sum_n a_n \phi_n(z) \right\}$$

$\mathbb{C}^{\mathbb{N}}$ is the space of sequence $\{a_n\}: \mathbb{N} \to \mathbb{C}$. The mapping α is defined as

$$\alpha: F_\Phi \to \mathbb{C}^{\mathbb{N}}, \quad f \mapsto \{a_n\}, \quad a_n = \lim_{z \to z_0} \frac{f(z) - \sum_{k=0}^{n-1} a_k \phi_k(z)}{\phi_n(z)}$$

Then α is surjective.

Proof. For given $\{\phi_n\}$ and $\{a_n\}$, we want to construct a function f that has the corresponding asymptotic expansion. If $a_k = 0$, then we can reject ϕ_k. Without loss of generality, we can take all $a_n \neq 0$. Since $\phi_n = o(\phi_{n-1})$, we have $a_n \phi_n = o(a_{n-1}\phi_{n-1})$. This can be shown as

$$\forall \varepsilon = \tilde{\varepsilon} \frac{|a_{n-1}|}{|a_n|}, \quad \exists \delta \quad s.t. \quad |\phi_n(z)| \leq \varepsilon |\phi_{n-1}(z)|, \quad |a_n \phi_n(z)| \leq \tilde{\varepsilon}\, |a_{n-1}\phi_{n-1}(z)|$$

Take $\varepsilon = 1/2$, and there exists a decreasing sequence $\{r_n\}$ or positive radii $r_n > 0$ such that

$$|a_n \phi_n(z)| \leq \frac{1}{2} |a_{n-1}\phi_{n-1}(z)|, \qquad z \in \bar{B}(z_0, r_n) \cap \Omega$$

Now introduce a sequence of cut-off functions $\{\mu_n(z)\}$ defined as follows

$$\mu_n(z) = \begin{cases} 1, & |z - z_0| \leq r_{n+1} \\ \text{linear of smooth}, & r_{n+1} < |z - z_0| \leq r_n \\ 0, & |z - z_0| > r_n \end{cases}$$

We construct the function $f(z)$ as

$$f(z) = \sum_{n \geq 0} a_n \mu_n(z) \phi_n(z), \qquad z \in \Omega$$

For $z \in \Omega$, we can find N such that $|z - z_0| > r_N$. We can see that the series is convergent

$$f(z) = \sum_{n=0}^{N} a_n \mu_n(z) \phi_n(z)$$

Because $\{r_n\}$ is decreasing, then $\mu_n(z) \leq \mu_{n-1}(z)$. Also, note that

$$|a_n \phi_n(z)| \leq \frac{1}{2} |a_{n-1}\phi_{n-1}(z)|, \qquad \text{when } \mu_n(z) \neq 0$$

When $|z - z_0| \leq r_{N+1}$, the remainder can be written as

$$R_N(z) = f(z) - \sum_{n=0}^{N} a_n \phi_n(z) = f(z) - \sum_{n=0}^{N} a_n \mu_n(z) \phi_n(z) = \sum_{n=N+1}^{\infty} a_n \mu_n(z) \phi_n(z)$$

We want to prove $R_N = o(\phi_N)$ as $z \to z_0$.

$$|R_N(z)| \leq \sum_{n=N+1}^{\infty} \mu_n(z)|a_n\phi_n(z)| \leq \mu_{N+1}(z)|a_{N+1}\phi_{N+1}(z)| + \sum_{n=N+2}^{\infty} \mu_n(z)|a_n\phi_n(z)|$$

The second term can be manipulated as

$$\mu_n(z)|a_n\phi_n(z)| \leq \frac{1}{2}\mu_n(z)|a_{n-1}\phi_{n-1}(z)| \leq \frac{1}{2}\mu_{n-1}(z)|a_{n-1}\phi_{n-1}(z)|$$

This leads to

$$\sum_{n=N+1}^{\infty} \mu_n(z)|a_n\phi_n(z)| \leq \mu_{N+1}(z)|a_{N+1}\phi_{N+1}(z)| + \frac{1}{2}\sum_{n=N+1}^{\infty} \mu_n(z)|a_n\phi_n(z)|$$

$$|R_N(z)| \leq 2\mu_{N+1}(z)|a_{N+1}\phi_{N+1}(z)| \leq 2|a_{N+1}\phi_{N+1}(z)|$$

As $\phi_{N+1} = o(\phi_N)$, we prove $R_N = o(\phi_N)$ and it is the asymptotic expansion of $f(z)$. ■

A similar result is **Borel's Lemma**, which states that $\forall\{a_n\}, \exists f \in C^\infty$ such that $f^{(n)}(0) = a_n$.

- Asymptotic root finding (1.5)

Consider polynomials $f_i(x)$, and we want to find the root of the following equation

$$f(x,\varepsilon) = f_0(x) + \varepsilon f_1(x) = 0$$

First we need the following implicit function theorem.

Theorem. For domains $\Omega, \mathrm{D} \in \mathbb{C}$ with $0 \in D$, consider a holomorphic function $f \in O(\Omega \times D)$. For a point $x_0 \in \Omega$, if $f(x_0, 0) = 0$ and $f_x(x_0, 0) \neq 0$, then there exists an open subset $D' \subseteq D$ with $0 \in D'$ and an analytic $\phi: D' \to \Omega$ such that $\forall \varepsilon \in D'$, $f(\phi(\varepsilon), \varepsilon) = 0$ and $\phi(0) = x_0$.

If $\deg(f_1) < \deg(f_0)$, then denote $d = \deg(f_0) = \deg(f)$ and f_0 has d roots. We can infer that x_0 is a simple root of f_0 since

$$\frac{\partial f}{\partial x}(x_0, 0) = \frac{\partial f_0}{\partial x}(x_0) \neq 0$$

Example 1: Regular perturbation problem with $\deg(f_1) < \deg(f_0)$

$$f(x,\varepsilon) = x^3 - x + \varepsilon = 0, \qquad f_0(x) = x^3 - x, \qquad f_1(x) = 1$$

The roots of $f_0(x)$ are $x_{1,2,3} = 0, -1, +1$. For each specific x_i, consider

$$x = x_i + \varepsilon x_i^{(1)} + \varepsilon^2 x_i^{(2)} + \cdots + \varepsilon^N x_i^{(N)} + o(\varepsilon^N)$$

Then we have

$$f(x,\varepsilon) = \left(x_i + \varepsilon x_i^{(1)} + \cdots\right)^3 - \left(x_i + \varepsilon x_i^{(1)} + \cdots\right) + \varepsilon = 0$$

Comparing the coefficients for each order of ε gives

$$[\varepsilon^0]:\ x_i^3 - x_i = 0$$

$$[\varepsilon^1]:\ 3x_i^2 x_i^{(1)} - x_i^{(1)} + 1 = 0, \qquad x_i^{(1)} = \frac{1}{1-3x_i^2}$$

$$[\varepsilon^2]:\ 3x_i\left[x_i^{(1)}\right]^2 + 3x_i^2 x_i^{(2)} - x_i^{(2)} = 0, \qquad x_i^{(2)} = \frac{3x_i}{(1-3x_i^2)^3}$$

Example 2: Singular perturbation problem with $\deg(f_1) > \deg(f_0)$

$$f(x,\varepsilon) = \varepsilon x^d - x + 1 = 0, \qquad d \geq 2$$

When $\varepsilon = 0$, we only have one root $x = 1$. We want to understand the other $d-1$ roots. As a simple case, consider $d = 2$ and we have

$$x_{1,2} = \frac{1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}, \qquad \sqrt{1-4\varepsilon} = 1 - 2\varepsilon - 2\varepsilon^2 + o(\varepsilon^2)$$

Then we can write

$$x_1 = \frac{1}{\varepsilon} - 1 - \varepsilon + o(\varepsilon), \qquad x_2 = 1 + \varepsilon + o(\varepsilon)$$

As $\varepsilon \to 0$, we have $x_1 \to \infty$ and thus it disappears when solving $f(x,0) = 0$. We consider roots in the form $x = \varepsilon^{-\rho} y$ with $\rho > 0$, such that $f(x,\varepsilon) = g(y, \varepsilon^{\rho})$ becomes a regular perturbation problem as follows

$$\varepsilon \cdot \varepsilon^{-d\rho} y^d - \varepsilon^{-\rho} y + 1 = 0, \qquad y^d - \varepsilon^{d\rho-\rho-1} y + \varepsilon^{d\rho-1} = 0$$

Therefore, we can choose

$$d\rho - \rho - 1 = 0, \qquad \rho = \frac{1}{d-1}, \qquad y^d - y + \varepsilon^{\rho} = 0$$

Denote $\delta = \varepsilon$^ρ. We can first solve for $y_i(\delta)$ and then obtain $x_i = \delta^{-1} y_i(\delta)$.

Example 3: Principle of dominant balance

$$f(x,\varepsilon) = \varepsilon x^3 + \varepsilon x^2 - x + 1 = 0$$

Following the same procedure, consider $x = \varepsilon^{-\rho} y$ with $\rho > 0$, and we obtain

$$\varepsilon^{1-3\rho} y^3 + \varepsilon^{1-2\rho} y^2 - \varepsilon^{-\rho} y + 1 = 0$$

For the four terms, we notice $1 - 3\rho < 1 - 2\rho$ and $-\rho < 0$. This implies that terms I and III are dominant, and we set $1 - 3\rho = -\rho$, which gives $\rho = 1/2$. The equation becomes

$$y^3 + \varepsilon^{1/2} y^2 - y + \varepsilon^{1/2} = 0$$

From the algebraic perspective, the techniques for solving singular perturbation problems are essentially proving that the field of Puiseux series is the algebraic closure of the field of formal Laurent series over the complex domain $\mathbb{C}$.

- Exercise

Asymptotic root finding

$$\varepsilon x^3 = (x-1)^2$$

When $\varepsilon = 0$, the root $x_1 = x_2 = 1$ are not simple roots. To obtain the perturbation around these roots, consider

$$\pm\sqrt{\varepsilon}\, x^{3/2} = x - 1, \qquad \tilde{x}_i = x_i + \sqrt{\varepsilon}\, x_i^{(1)} + \varepsilon x_i^{(2)} + \cdots$$

We have $x_{1,2} = 1$ and

$$\tilde{x}_i^{3/2} = 1 + \frac{3}{2}\left(\sqrt{\varepsilon}\, x_i^{(1)} + \varepsilon x_i^{(2)} + \cdots\right) + \frac{3}{8}\left(\sqrt{\varepsilon}\, x_i^{(1)} + \varepsilon x_i^{(2)} + \cdots\right)^2 + \cdots$$

By comparing the coefficients, we obtain

$$\tilde{x}_{1,2} = 1 \pm \sqrt{\varepsilon} + \frac{3}{2}\varepsilon \pm \frac{21}{8}\varepsilon^{3/2} + 5\varepsilon^2 + O\left(\varepsilon^{5/2}\right), \qquad \varepsilon \to 0^+$$

Another root $\tilde{x}_3$ goes to infinity as $\varepsilon \to 0^+$. Consider $x = \varepsilon^{-\rho} y$ with $\rho > 0$, and we obtain

$$\varepsilon^{1-3\rho} y^3 = \varepsilon^{-2\rho} y^2 - 2\varepsilon^{-\rho} y + 1$$

From the dominant balance, we set $1 - 3\rho = -2\rho$, which gives $\rho = 1$ and

$$y^3 - y^2 + 2\varepsilon y - \varepsilon^2 = 0$$

We need to take the root $y_0 = 1$ when $\varepsilon = 0$ to ensure y is bounded away from zero.

$$\tilde{y} = 1 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \varepsilon^4 y_4 + \cdots$$

By comparing the coefficients, we obtain

$$\tilde{y} = 1 - 2\varepsilon - 3\varepsilon^2 - 10\varepsilon^3 - 42\varepsilon^4 + \cdots$$

Then the third root is given as

$$\tilde{x}_3 = \frac{1}{\varepsilon} - 2 - 3\varepsilon - 10\varepsilon^2 - 42\varepsilon^3 + O(\varepsilon^4)$$

Asymptotic Analysis of Integrals (1): Watson's Lemma

We will study the following four types of exponential integrals with corresponding methods.

$\int_0^T e^{-\lambda t}\phi(t)\,\mathrm{d}t$	Watson's Lemma
$\int_a^b e^{\lambda R(t)}g(t)\,\mathrm{d}t$	Laplace's method
$\int_\gamma e^{-\lambda h(z)}g(z)\,\mathrm{d}z$	Method of steepest descent
$\int_a^b e^{i\lambda I(t)}g(t)\,\mathrm{d}t$	Method of stationary phase

In this chapter, we will discuss the first type of integral.

Example: Incomplete Gamma function (2.1)

$$\gamma(z,x)=\int_0^x e^{-t}t^{z-1}\,\mathrm{d}t\,,\qquad x>0,\ \ z>0$$

1. As $x\to 0^+$, with Taylor expansion we have

$$e^{-t}=\sum_{n=0}^{N}\frac{(-1)^n}{n!}t^n+R_N(t),\qquad R_N(t)=\frac{(-1)^{N+1}}{N!}\int_0^t e^{-s}(t-s)^N\,\mathrm{d}s$$

The remainder term can be bounded as

$$|R_N(t)|\le\frac{1}{N!}\int_0^t (t-s)^N\,\mathrm{d}s=\frac{t^{N+1}}{(N+1)!}=c_1t^{N+1}$$

The integral becomes

$$\gamma(z,x)=\int_0^x\left(\sum_{n=0}^{N}\frac{(-1)^n}{n!}t^n+R_N(t)\right)t^{z-1}\,\mathrm{d}t=\sum_{n=0}^{N}\frac{(-1)^n}{n!}\frac{x^{n+z}}{n+z}+o(x^{N+z})$$

Therefore, the asymptotic expansion of $\gamma(z,x)$ at $x=0$ is

$$\gamma(z,x)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{n+z}}{n+z},\qquad x\to 0^+$$

2. As $x\to\infty$, note that

$$\gamma(x,z)=\Gamma(z)-\int_x^\infty e^{-t}t^{z-1}\,\mathrm{d}t$$

In this case, e^{-t} is dominant for the convergence. Denote $t=x+s$ and we have

$$\int_x^\infty e^{-t}t^{z-1}\,\mathrm{d}t=e^{-x}x^{z-1}\int_0^\infty e^{-s}\left(1+\frac{s}{x}\right)^{z-1}\mathrm{d}s$$

Now we expand $(1+s/x)^{z-1}$ as $x\to\infty$, which is

$$\left(1+\frac{s}{x}\right)^{z-1}=\sum_{n=0}^{N}\binom{z-1}{n}\left(\frac{s}{x}\right)^{n}+R_N(s),\qquad R_N(s)=O\left(\frac{s^{N+1}}{x^{N+1}}\right)$$

The integral becomes

$$\begin{aligned}\int_x^\infty e^{-t}t^{z-1}\,\mathrm{d}t&=e^{-x}x^{z-1}\left[\sum_{n=0}^{N}\binom{z-1}{n}\frac{1}{x^n}\int_0^\infty e^{-s}s^n\,\mathrm{d}s+O\left(\frac{1}{x^{N+1}}\int_0^\infty e^{-s}s^{N+1}\,\mathrm{d}s\right)\right]\\&=e^{-x}x^{z-1}\left[\sum_{n=0}^{N}\frac{(z-1)\cdots(z-n)}{x^n}+o\left(\frac{1}{x^N}\right)\right]\end{aligned}$$

Therefore, the asymptotic expansion of $\gamma(z,x)$ at $x\to\infty$ is

$$\gamma(z,x)\sim\Gamma(z)-\sum_{n=0}^{\infty}(z-1)\cdots(z-n)\cdot e^{-x}x^{z-n-1},\qquad x\to\infty$$

Another way to derive the result is to use integration by parts.

- Watson's Lemma (2.2)

Suppose $T>0$ and $\phi(t)\colon[0,T]\to\mathbb{C}$ is absolutely integrable

$$\int_0^T|\phi(t)|\,\mathrm{d}t<\infty$$

Further suppose that there exists $\sigma>-1$ and $g(t)\in C^\infty$ near $t=0^+$ such that $\phi(t)=t^\sigma g(t)$. Then the exponential integral is finite for all $\lambda>0$, with the following asymptotic expansion

$$F(\lambda)=\int_0^T e^{-\lambda t}\phi(t)\,\mathrm{d}t\sim\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}\cdot\frac{\Gamma(\sigma+n+1)}{\lambda^{\sigma+n+1}},\qquad\lambda\to\infty$$

Proof. Choose $s>0$ arbitrarily, and we have

$$\left|\int_s^T e^{-\lambda t}\phi(t)\,\mathrm{d}t\right|\le e^{-\lambda s}\int_s^T|\phi(t)|\,\mathrm{d}t\le Me^{-\lambda s}=o\big(\lambda^{-(\sigma+n+1)}\big),\qquad\forall n\in\mathbb{N}$$

Therefore, we can split up the integral and only focus on $[0,s]$. For a sufficiently small $s>0$ such that $g\in C^\infty(0,s)$ and $\phi(t)=t^\sigma g(t)$. The Taylor expansion of $g(t)$ is

$$g(t)=\sum_{n=0}^{N}\frac{1}{n!}g^{(n)}(0)t^n+R_N(t),\qquad R_N(t)=O(t^{N+1})$$

The integral becomes

$$\int_0^s e^{-\lambda t}\phi(t)\,\mathrm{d}t=\sum_{n=0}^{N}\frac{g^{(n)}(0)}{n!}\int_0^s e^{-\lambda t}t^{\sigma+n}\,\mathrm{d}t+O\left(\int_0^s e^{-\lambda t}t^{\sigma+N+1}\,\mathrm{d}t\right)$$

Denote the integral that appears as $F_p(\lambda)$ for each integer $p\ge0$. With $x=\lambda t$, we have

$$F_p(\lambda) = \int_0^s e^{-\lambda t} t^{\sigma+p}\,\mathrm{d}t = \int_0^{\lambda s} e^{-x}\left(\frac{x}{\lambda}\right)^{\sigma+p}\frac{\mathrm{d}x}{\lambda} = \frac{1}{\lambda^{\sigma+p+1}}\int_0^{\lambda s} e^{-x}x^{\sigma+p}\,\mathrm{d}x$$

$$= \frac{1}{\lambda^{\sigma+p+1}}\left[\Gamma(\sigma+p+1) - \int_{\lambda s}^{\infty} e^{-x}x^{\sigma+p}\,\mathrm{d}x\right]$$

One trick to manipulate the second term is to use the Cauchy inequality.

$$\int_{\lambda s}^{\infty} e^{-x}x^{\sigma+p}\,\mathrm{d}x = \int_{\lambda s}^{\infty} e^{-\frac{x}{2}}\cdot e^{-\frac{x}{2}}x^{\sigma+p}\,\mathrm{d}x \le \sqrt{\int_{\lambda s}^{\infty}\left(e^{-\frac{x}{2}}\right)^2\mathrm{d}x}\cdot\sqrt{\int_{\lambda s}^{\infty}\left(e^{-\frac{x}{2}}x^{\sigma+p}\right)^2\mathrm{d}x}$$

$$= e^{-\lambda s/2}\cdot\sqrt{\int_{\lambda s}^{\infty} e^{-x}x^{2(\sigma+p)}\,\mathrm{d}x} \le e^{-\lambda s/2}\cdot\sqrt{\int_{s}^{\infty} e^{-x}x^{2(\sigma+p)}\,\mathrm{d}x}$$

Since λ is very large, we can reduce the lower bound of the integral and remove its dependence on λ. It can also be written as $o(\lambda^{-\sigma-n-1})$ for any $n \in \mathbb{N}$. Eventually, we have

$$F(\lambda) \sim \sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}\cdot\frac{\Gamma(\sigma+n+1)}{\lambda^{\sigma+n+1}}, \qquad \lambda\to\infty$$

Generalizations of Watson's Lemma (2.3)

- Replace the condition as $\lambda \in \Omega_\theta = \{z \in \mathbb{C} \mid |\arg z| < \theta\}$, and consider $|\lambda| \to \infty$.
- Replace the condition as $\mathrm{Re}(\sigma) > -1$.
- The function g can be only finitely differentiable as $g \in C^m$.
- If $T \to +\infty$, then we only need to require $e^{-\lambda t}\phi(t)$ to be absolutely integrable.

Example 1

$$F(\lambda) = \int_0^{\infty} e^{-\lambda t}\ln(1+t^2)\,\mathrm{d}t$$

Similarly choose $s > 0$ arbitrarily, and from Cauchy inequality we have

$$\int_s^{\infty} e^{-\lambda t}\ln(1+t^2)\,\mathrm{d}t \le \sqrt{\int_s^{\infty} t^2 e^{-2\lambda t}\,\mathrm{d}t}\cdot\sqrt{\int_s^{\infty}\frac{\ln^2(1+t^2)}{t^2}\,\mathrm{d}t} = o(\lambda^{-\sigma-n-1})$$

This is the only step we need to modify from the previous proof. With the Taylor expansion of $\phi(t) = \ln(1+t^2)$, we have

$$F(\lambda) \sim \int_0^{\infty} e^{-\lambda t}\left[\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{2n}}{n}\right]\mathrm{d}t = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{(2n)!}{\lambda^{2n+1}}, \qquad \lambda\to\infty$$

Although the series is not convergent, in the sense of asymptotic analysis it is valid. Therefore, after truncating the sum, the result makes sense.

Example 2

$$F(\lambda) = \int_{-\alpha}^{\beta} e^{-\lambda t^2} \phi(t)\,\mathrm{d}t\,, \qquad \alpha, \beta > 0$$

We consider $\phi(t) \in C^\infty$ around $t = 0$. For a sufficiently small $s > 0$, note that

$$\left|\int_s^{\beta} e^{-\lambda t^2} \phi(t)\,\mathrm{d}t\right| \le e^{-\lambda s^2} \int_{s^2}^{\beta^2} \left|\frac{\phi(\sqrt{u})}{2\sqrt{u}}\right| \mathrm{d}u = o(\lambda^p), \qquad \forall p \in \mathbb{R}$$

This implies that we only need to analyze the symmetric part, which is

$$F(\lambda) \sim \int_{-s}^{s} e^{-\lambda t^2} \phi(t)\,\mathrm{d}t = 2\int_0^s e^{-\lambda t^2} \left[\sum_{n=0}^{N} \frac{\phi^{(2n)}(0)}{(2n)!} t^{2n} + o(t^{2N})\right] \mathrm{d}t$$

Evaluating the integral, we obtain

$$F(\lambda) \sim \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{(2n)!} \cdot \frac{\Gamma(n + 1/2)}{\lambda^{n+1/2}} = \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n}\, n!} \cdot \frac{1}{\lambda^n}, \qquad \lambda \to \infty$$

- Exercise

Term-by-term integration

$$\int_0^{\infty} e^{-t}(1 + zt)^{-1}\,\mathrm{d}t\,, \qquad z \to 0, \qquad |\arg z| < \frac{\pi}{2}$$

Expanding $(1 + zt)^{-1}$ as $z \to 0$ gives

$$(1 + zt)^{-1} = \sum_{n=0}^{N} (-1)^n z^n t^n + R_N(t), \qquad R_N(t) = O(z^{N+1} t^{N+1})$$

The integral thus becomes

$$\begin{aligned} \int_0^{\infty} e^{-t}(1 + zt)^{-1}\,\mathrm{d}t &= \sum_{n=0}^{N} (-1)^n z^n \int_0^{\infty} e^{-t} t^n\,\mathrm{d}t + O\left(z^{N+1} \int_0^{\infty} e^{-t} t^{N+1}\,\mathrm{d}t\right) \\ &= \sum_{n=0}^{N} (-1)^n \cdot n!\, z^n + o(z^N) \end{aligned}$$

Therefore, the asymptotic expansion is

$$\int_0^{\infty} e^{-t}(1 + zt)^{-1}\,\mathrm{d}t \sim \sum_{n=0}^{\infty} (-1)^n \cdot n!\, z^n\,, \qquad z \to 0, \qquad |\arg z| < \frac{\pi}{2}$$

Exponential integral

$$e^{-x} \int_0^{\infty} \frac{e^{-t}}{x + t}\,\mathrm{d}t\,, \qquad x \to \infty$$

Similarly, by expanding $(1+t/x)^{-1}$ as $x\to\infty$, we have

$$\begin{aligned} e^{-x}\int_0^\infty \frac{e^{-t}}{x+t}\mathrm{d}t &= \frac{e^{-x}}{x}\int_0^\infty \frac{e^{-t}}{1+\frac{t}{x}}\mathrm{d}t \\ &= \frac{e^{-x}}{x}\left[\sum_{n=0}^{N}\frac{(-1)^n}{x^n}\int_0^\infty e^{-t}t^n\,\mathrm{d}t + O\left(\frac{1}{x^{N+1}}\int_0^\infty e^{-t}t^{N+1}\,\mathrm{d}t\right)\right] \\ &= e^{-x}\sum_{n=0}^{N}\frac{(-1)^n\cdot n!}{x^{n+1}} + o\left(\frac{1}{x^{N+1}}\right) \end{aligned}$$

The asymptotic expansion of the integral, which is also for the function $\mathrm{Ei}(x)$, is

$$e^{-x}\int_0^\infty \frac{e^{-t}}{x+t}\mathrm{d}t \sim e^{-x}\sum_{n=0}^{\infty}\frac{(-1)^n\cdot n!}{x^{n+1}},\qquad x\to\infty$$

Error function

$$\operatorname{erf}x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t,\qquad x\in\mathbb{R}$$

1. As $x\to 0^+$, with Taylor expansion we have

$$e^{-t^2} = \sum_{n=0}^{N}\frac{(-1)^n}{n!}t^{2n} + R_N(t),\qquad R_N(t) = \frac{(-1)^{N+1}}{N!}\int_0^t 2e^{-s^2}(t-s)^{2N+1}\,\mathrm{d}s$$

The remainder term can be bounded as

$$|R_N(t)| \le \frac{2}{N!}\int_0^t (t-s)^{2N+1}\,\mathrm{d}s = \frac{t^{2N+2}}{(N+1)!} = c_1 t^{2N+2}$$

The integral becomes

$$\operatorname{erf}x = \frac{2}{\sqrt{\pi}}\int_0^x\left(\sum_{n=0}^{N}\frac{(-1)^n}{n!}t^{2n} + R_N(t)\right)\mathrm{d}t = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{N}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1} + o(x^{2N+2})$$

Therefore, the asymptotic expansion of $\operatorname{erf}x$ at $x=0$ is

$$\operatorname{erf}x \sim \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1},\qquad x\to 0^+$$

2. As $x\to\infty$, note that

$$\operatorname{erf}x = 1-\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\mathrm{d}t = 1-\operatorname{erfc}x$$

With the change of variable $u+x^2=t^2$, we have

$$\operatorname{erfc}x = \frac{e^{-x^2}}{x\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{1+u/x^2}}\,\mathrm{d}u$$

Now we expand $(1+u/x^2)^{-1/2}$ as $x\to\infty$, which is

$$\left(1+\frac{u}{x^2}\right)^{-1/2} = \sum_{n=0}^{N}(-1)^n \frac{(2n-1)!!}{n!\cdot(2x^2)^n}u^n + R_N(u), \qquad R_N(u) = O\left(\frac{u^{N+1}}{x^{2N+2}}\right)$$

The integral becomes

$$\operatorname{erfc} x = \frac{e^{-x^2}}{x\sqrt{\pi}}\left[\sum_{n=0}^{N}(-1)^n \frac{(2n-1)!!}{n!\cdot(2x^2)^n}\int_0^\infty e^{-u}u^n \,\mathrm{du} + O\left(\frac{1}{x^{2N+2}}\int_0^\infty e^{-u}u^{N+1}\,\mathrm{du}\right)\right]$$

$$= \frac{e^{-x^2}}{x\sqrt{\pi}}\left[\sum_{n=0}^{N}(-1)^n \frac{(2n-1)!!}{(2x^2)^n} + o\left(\frac{1}{x^{2N+1}}\right)\right]$$

Therefore, the asymptotic expansion of $\operatorname{erf} x$ at $x \to \infty$ is

$$\operatorname{erf} x \sim 1 - \frac{e^{-x^2}}{x\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n \frac{(2n-1)!!}{(2x^2)^n}, \qquad x \to \infty$$

Integral with e^{-xt^2} kernel

$$\int_0^\infty e^{-xt^2}\sin t\,\mathrm{d}t\,, \qquad x \to \infty$$

Denote $\phi(t) = \sin t$, and its Taylor series is

$$\sin t = a_1 t + a_3 t^3 + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

From the previous result, as $x \to \infty$ we have

$$\int_0^\infty e^{-xt^2}\sin t\,\mathrm{d}t \sim \frac{1}{2}\int_0^\infty e^{-xu}\left(a_1 + a_3 u + \cdots\right)\mathrm{du} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^\infty e^{-xu}\,u^n\,\mathrm{du}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{x^{n+1}}\int_0^\infty e^{-u}\,u^n\,\mathrm{du} = \frac{1}{2x}\sum_{n=0}^{\infty}(-1)^n\frac{n!}{(2n+1)!}\frac{1}{x^n}$$

The first few terms in the asymptotic expansion is

$$\int_0^\infty e^{-xt^2}\sin t\,\mathrm{d}t \sim \frac{1}{2x}\left(1 - \frac{1}{6x} + \frac{1}{60x^2} - \frac{1}{840x^3} + \cdots\right), \qquad x \to \infty$$

Integral with e^{-xt^3} kernel

$$\int_0^1 e^{-xt^3}\mathrm{d}t\,, \qquad x \to \infty$$

With the change of variable $u = xt^3$, we have

$$\int_0^1 e^{-xt^3}\mathrm{d}t \sim \frac{1}{3x^{1/3}}\int_0^x e^{-u}\,u^{-\frac{2}{3}}\,\mathrm{du} = \frac{\Gamma(1/3)}{3x^{1/3}} - \frac{e^{-x}}{3x}\left(1 + O(x^{-1})\right), \qquad x \to \infty$$

Asymptotic Analysis of Integrals (2): Laplace's Method

We analyze the asymptotic expansion of the following integral

$$F(\lambda)=\int_a^b e^{\lambda R(t)}g(t)\,\mathrm{d}t\,,\qquad \lambda\to\infty$$

With $R(t)\in C[a,b]$ and $g(t)$ an absolutely integrable function.

➢ Nonlocal contributions (3.2)

We know that R can reach its maximum $R_{\max}$ within $[a,b]$. Denote $\bar{R}(t)=R(t)-R_{\max}$, then

$$F(\lambda)=e^{\lambda R_{\max}}\int_a^b e^{\lambda\bar{R}(t)}g(t)\,\mathrm{d}t\,,\qquad \max\bar{R}(t)=0$$

Without loss of generality, we can take $R_{\max}=0$. For the previous examples, we have

$$e^{-\lambda t},\qquad R(t)=-t,\qquad t\in[0,T],\qquad R_{\max}=0$$

$$e^{-\lambda t^2},\qquad R(t)=-t^2,\qquad t\in[-\alpha,\beta],\qquad R_{\max}=0$$

Define the set T as

$$T=\{t\in[a,b]\mid R(t)=0\}$$

We assume that T is finite. For $\delta>0$, define sets T_δ and I_δ as

$$T_\delta=\{t\in[a,b]\mid d(t,T)<\delta\},\qquad I_\delta=[a,b]\setminus T_\delta$$

Note that T_δ is open and I_δ is closed and compact. Then $R(t)$ has a maximal value in I_δ, which is denoted as $-K_\delta<0$. We can show that

$$\left|\int_{I_\delta}e^{\lambda R(t)}g(t)\,\mathrm{d}t\right|\le\int_{I_\delta}e^{\lambda R(t)}|g(t)|\,\mathrm{d}t\le e^{-\lambda K_\delta}\int_a^b|g(t)|\,\mathrm{d}t=o(\lambda^p),\qquad \forall p\in\mathbb{R}$$

➢ Contributions from endpoints (3.3)

Consider $T=\{a\}$ as the left endpoint. We assume $R,g\in C^\infty[a,a+\delta)$.

$$F(\lambda)\sim\int_a^{a+\delta}e^{\lambda R(t)}g(t)\,\mathrm{d}t=\int_0^\delta e^{\lambda R(a+\tau)}g(a+\tau)\,\mathrm{d}\tau$$

First we consider $R'(a)\neq0$, and since $R(a)$ is maximum we have $R'(a)<0$. We need to find a change of variable $\tau=\tau(s)$ such that $R(a+\tau)=-s$. Based on the inverse function theorem, we know that $\tau(s)$ exists because

$$\tau(0)=0,\qquad u(\tau,s)=R(a+\tau)+s,\qquad \left.\frac{\partial u}{\partial\tau}\right|_{(\tau,s)=(0,0)}=R'(a)\neq0$$

With this change of variable, we can directly apply Watson's Lemma with $\sigma=0$ and

$$\phi(s)=g\big(a+\tau(s)\big)\,\tau'(s)$$

This leads to the following asymptotic expansion

$$F(\lambda) \sim \int_0^{\delta'} e^{-\lambda s} g\big(a+\tau(s)\big)\,\tau'(s)\,\mathrm{d}s = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{\lambda^{n+1}}, \qquad \lambda \to \infty$$

The calculation of higher derivatives of $g(a+\tau(s))$ follows Faà di Bruno formula, while those for $\tau(s)$ follows the formula for the inverse function. In particular, we have

$$\phi(0) = g(a)\,\tau'(0) = -\frac{g(a)}{R'(a)}$$

The leading order term gives

$$F(\lambda) \sim -\frac{g(a)}{R'(a)} \cdot \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty$$

If $R'(a) = R''(a) = \cdots = R^{(k-1)}(a) = 0$ and $R^{(k)}(a) < 0$, then we have

$$R(a+\tau) = R(a) + \frac{1}{k!} R^{(k)}(a)\tau^k + o(\tau^k)$$

The change of variable $\tau = \tau(s)$ now should satisfy $R(a+\tau) = -s^k$.

If $T = \{b\}$ is the right endpoint, we have

$$F(\lambda) \sim \frac{g(b)}{R'(b)} \cdot \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty$$

- Contributions from interior maxima (3.4)

Consider $T = \{t_{\max}\}$ with $a < t_{\max} < b$. We have $R'(t_{\max}) = 0$ and assume $R''(t_{\max}) < 0$.

$$F(\lambda) \sim \int_{-\delta}^{\delta} e^{\lambda R(t_{\max}+\tau)} g(t_{\max}+\tau)\,\mathrm{d}\tau$$

Morse Lemma

For $f \in C^3(\Omega)$, $x_0 \in \Omega$ and a non-degenerate $D^2 f$, there exists a local coordinate transform $x = x(y)$ such that

$$f\big(x(y)\big) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2 + f(x_0)$$

From Morse Lemma, we can find a transform $\tau = \tau(s)$ such that

$$R - R_{\max} = \frac{1}{2} R''(t_{\max})(t - t_{\max})^2 + \cdots = -s^2$$

Take $R_{\max} = 0$ and this leads to

$$F(\lambda) \sim \int_{-\alpha}^{\beta} e^{-\lambda s^2} g\big(t_{\max}+\tau(s)\big)\,\tau'(s)\,\mathrm{d}s \sim \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n}\,n!} \cdot \frac{1}{\lambda^n}$$

Here we use the previous result with

$$\phi(s) = g\big(t_{\max} + \tau(s)\big)\,\tau'(s), \qquad \phi(0) = g(t_{\max})\,\tau'(0)$$

The derivative $\tau'(s)$ is obtained from

$$R\big(t_{\max} + \tau(s)\big) = -s^2, \qquad R'\big(t_{\max} + \tau(s)\big)\,\tau'(s) = -2s$$

$$R''\big(t_{\max} + \tau(s)\big)\,[\tau'(s)]^2 + R'\big(t_{\max} + \tau(s)\big)\,\tau''(s) = -2$$

At $s = 0$, we have

$$R''(t_{\max})\,[\tau'(0)]^2 = -2, \qquad \tau'(0) = \sqrt{-\frac{2}{R''(t_{\max})}}$$

The leading order term gives

$$F(\lambda) \sim \frac{g(t_{\max})}{\sqrt{|R''(t_{\max})|}}\sqrt{\frac{2\pi}{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right), \qquad \lambda \to \infty$$

The interior maxima dominates over the endpoints.

Example 1: Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t\,, \qquad z \to \infty$$

Consider a change of variable $t = zs$, we have

$$z\Gamma(z) = \int_0^\infty e^{-t} t^z\,\mathrm{d}t = \int_0^\infty e^{-zs + z(\ln z + \ln s)}\,z\mathrm{d}s = z \cdot z^z \int_0^\infty e^{z(\ln s - s)}\,\mathrm{d}s$$

We recognize $R(s) = \ln s - s$ with $s_{\max} = 1$ and $R(s_{\max}) = -1$. Now we have

$$\Gamma(z) = z^z e^{-z} \int_0^\infty e^{z(\ln s - s + 1)}\,\mathrm{d}s\,, \qquad \tilde{R}(s) = \ln s - s + 1$$

We want a coordinate transform $s = s(u)$ such that

$$\ln s - s + 1 = -u^2, \qquad -se^{-s} = -e^{-(1+u^2)}, \qquad s = -W\big(-e^{-(1+u^2)}\big)$$

Now s is expressed using the Lambert W function. The asymptotic expansion becomes

$$\Gamma(z) \sim z^z e^{-z} \int_{-\delta}^{\delta} e^{-zu^2} s'(u)\,\mathrm{d}u$$

To obtain each coefficient, consider

$$s = 1 + c_1 u + c_2 u^2 + c_3 u^3 + o(u^3)$$

We thus obtain

$$\ln(1 + c_1 u + c_2 u^2 + c_3 u^3 + \cdots) = c_1 u + (c_2 - 1)u^2 + c_3 u^3 + \cdots$$

$$c_1 = \sqrt{2}, \qquad c_2 = \frac{2}{3}, \qquad c_3 = \frac{1}{9\sqrt{2}}$$

$$s'(u) = \sqrt{2} + \frac{4}{3}u + \frac{1}{3\sqrt{2}}u^2 + o(u^2)$$

Directly calculating the integral over the entire real axis, we have

$$\Gamma(z) \sim z^z e^{-z}\sqrt{\frac{2\pi}{z}}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840}\frac{1}{z^3} + O\left(\frac{1}{z^4}\right)\right)$$

Example 2: Weakly diffusive regularization of shock waves (3.6)

The shock wave equation is given as

$$u_t + uu_x = 0, \qquad u(x,0) = u_0(x)$$

The solution can be written as

$$u(x,t) = u_0(x - u(x,t)t)$$

After finite time, the initial $u_0(x)$ profile will experience catastrophic steepening that develops a shock wave. By including the diffusion term, the shock is regularized.

$$u_t + uu_x = \nu u_{xx}, \qquad u(x,0) = u_0(x)$$

This is **Burgers' equation** with small $\nu \to 0^+$. With the **Cole-Hopf transformation**, we have

$$\varphi_t = \nu\varphi_{xx}, \qquad u = -2\nu\frac{\varphi_x}{\varphi}$$

This leads to

$$\begin{aligned} u_t + uu_x - \nu u_{xx} &= -2\nu(\ln\varphi)_{xt} + 4\nu^2(\ln\varphi)_x(\ln\varphi)_{xx} + 2\nu^2(\ln\varphi)_{xxx} \\ &= [-2\nu(\ln\varphi)_t + 2\nu^2[(\ln\varphi)_x]^2 + 2\nu^2(\ln\varphi)_{xx}]_x \\ &= \left[-2\nu\frac{\varphi_t}{\varphi} + 2\nu^2\frac{\varphi_{xx}}{\varphi}\right]_x = \left[-2\nu\frac{\varphi_t - \nu\varphi_{xx}}{\varphi}\right]_x = 0 \end{aligned}$$

Now we can choose $\varphi_0(x)$ as

$$u_0(x) = -2\nu\frac{\varphi_0'}{\varphi_0}, \qquad \varphi_0(x) = \exp\left(-\frac{1}{2\nu}\int_0^x u_0(\eta)\,\mathrm{d}\eta\right)$$

The diffusion equation gives

$$\begin{aligned} \varphi(x,t) &= \frac{1}{\sqrt{4\pi\nu t}}\int_{\mathbb{R}} \varphi_0(\xi)\exp\left(-\frac{(x-\xi)^2}{4\nu t}\right)\mathrm{d}\xi \\ &= \frac{1}{\sqrt{4\pi\nu t}}\int_{\mathbb{R}} \exp\left(-\frac{1}{2\nu}\int_0^\xi u_0(\eta)\,\mathrm{d}\eta - \frac{(x-\xi)^2}{4\nu t}\right)\mathrm{d}\xi \end{aligned}$$

From the Cole-Hopf transformation, we can obtain

$$u(x,t) = \int_{\mathbb{R}} e^{\lambda R(\xi)} g(\xi)\,\mathrm{d}\xi \cdot \left[\int_{\mathbb{R}} e^{\lambda R(\xi)}\,\mathrm{d}\xi\right]^{-1}$$

The parameter λ and functions $R(\xi)$ and $g(\xi)$ correspond to

$$\lambda = \frac{1}{2\nu}, \qquad R(\xi; x, t) = -\int_0^{\xi} u_0(\eta)\, \mathrm{d}\eta - \frac{(x-\xi)^2}{2t}, \qquad g(\xi; x, t) = \frac{x-\xi}{t}$$

We first find the interior maxima of $R(\xi)$, which gives

$$R'(\xi) = -u_0(\xi) + \frac{x-\xi}{t} = 0, \qquad \xi(x,t) = x - tu_0(\xi)$$

From the Laplace's method, we have

$$u(x,t) = \frac{g(\xi) + O(\nu)}{1 + O(\nu)} = \frac{x-\xi}{t} + O(\nu), \qquad \nu \to 0$$

Taking the limit $\nu = 0$, we can recover the solution of the unperturbed problem as

$$x - tu = \xi = x - tu_0(x - tu), \qquad u(x,t) = u_0(x - u(x,t)t)$$

Now consider a matrix $\boldsymbol{A}(\boldsymbol{u})$ with real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_n), \qquad \boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_x = \boldsymbol{0}, \qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0$$

With the diffusion term, we have

$$\boldsymbol{u}_t + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u}_x = \varepsilon \boldsymbol{u}_{xx}, \qquad \boldsymbol{u}(x,0) = \boldsymbol{u}_0$$

This vector problem becomes much more difficult.

- Multidimensional integrals (3.7)

$$F(\lambda) = \int_\Omega e^{\lambda R(x)} g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{x} \in \mathbb{R}^n$$

Suppose that $R \in C^\infty$ takes its maximum at $\boldsymbol{x}_{\max} = \boldsymbol{0}$ in the interior of Ω with $R_{\max} = 0$.

$$R(\boldsymbol{x}) = R_{\max} + \frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{H}\boldsymbol{x} + o(|\boldsymbol{x}|^2), \qquad \boldsymbol{H} = \frac{\partial^2 R}{\partial x_i \partial x_j}(\boldsymbol{x}_{\max})$$

We assume that $\boldsymbol{H}$ is negative definite. The case of semi-definite $\boldsymbol{H}$ is more complicated.

Note: If $\boldsymbol{H} = \boldsymbol{0}$, then we need to consider $R^{(4)}(\boldsymbol{0})$. As a simple case, for $n = 2$ we have

$$R(x,y) = a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4 + o(|x^2+y^2|^2)$$

If $x \neq 0$, we have

$$R(x,y) = x^4\left[1 + a_2\frac{y}{x} + \cdots a_5\left(\frac{y}{x}\right)^4\right] + o(|x^2+y^2|^2)$$

To require a polynomial with $\deg R = 4$ always being negative is extremely complicated. This leads to the theory of singular points.

The integral can be similarly localized as

$$F(\lambda) \sim \int_{|x|\le\delta} e^{\lambda R(x)} g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

Based on Morse Lemma, there exists new coordinates $y_1, \cdots, y_n$ such that

$$R(\boldsymbol{x}(\boldsymbol{y})) = -y_1^n - \cdots - y_n^2 = -\boldsymbol{y}^T\boldsymbol{y}$$

Therefore, the integral becomes

$$F(\lambda) \sim \int_{|\boldsymbol{y}|\le\delta'} e^{-\lambda(y_1^2+\cdots+y_n^2)} g(\boldsymbol{x}(\boldsymbol{y}))\, J(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

Assume that all functions are C^∞. The Taylor expansion gives

$$\phi(\boldsymbol{y}) = \sum_{m\ge 0} \sum_{1\le k_1,\cdots,k_m\le n} C_{k_1\cdots k_m}\, y_{k_1}\cdots y_{k_m}$$

Here m denotes the polynomial degree. We need to evaluate the following type of integral

$$\int_{\mathbb{R}^n} e^{-\lambda(y_1^2+\cdots+y_n^2)}\, y_{k_1}\cdots y_{k_m}\, \mathrm{d}y_1\cdots\mathrm{d}y_n$$

Consider the function

$$\begin{aligned} G(x_1,\cdots,x_n) &= \int_{\mathbb{R}^n} e^{-\lambda(y_1^2+\cdots+y_n^2)+(x_1y_1+\cdots+x_ny_n)}\, \mathrm{d}y_1\cdots\mathrm{d}y_n \\ &= \left(\int_{\mathbb{R}^n} e^{-\lambda y_1^2+x_1y_1}\, \mathrm{d}y_1\right)\cdots\left(\int_{\mathbb{R}^n} e^{-\lambda y_n^2+x_ny_n}\, \mathrm{d}y_n\right) = \left(\frac{\pi}{\lambda}\right)^{\frac{n}{2}} \exp\left(\frac{x_1^2+\cdots+x_n^2}{4\lambda}\right) \end{aligned}$$

Then we have

$$\left.\frac{\partial^m G(x_1,\cdots,x_n)}{\partial x_{k_1}\cdots\partial x_{k_m}}\right|_{\boldsymbol{x}=\boldsymbol{0}} = \int_{\mathbb{R}^n} e^{-\lambda(y_1^2+\cdots+y_n^2)}\, y_{k_1}\cdots y_{k_m}\, \mathrm{d}y_1\cdots\mathrm{d}y_n$$

The integral is thus asymptotic to

$$F(\lambda) \sim \left(\frac{\pi}{\lambda}\right)^{n/2} \left[g(\boldsymbol{0})J(\boldsymbol{0}) + o\left(\frac{1}{\lambda}\right)\right]$$

To obtain the Jacobian $J(\boldsymbol{0})$, note that $\boldsymbol{x} = \boldsymbol{K}\boldsymbol{y}$ and we have

$$R(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{K}^T\boldsymbol{H}\boldsymbol{K}\boldsymbol{y} = -\boldsymbol{y}^T\boldsymbol{y}, \qquad \frac{1}{2}\boldsymbol{K}^T\boldsymbol{H}\boldsymbol{K} = -\boldsymbol{I}$$

From definition, we have $J = |\det \boldsymbol{K}|$, which is obtained as

$$\frac{1}{2^n}|\det \boldsymbol{K}|^2|\det \boldsymbol{H}| = 1, \qquad J = |\det \boldsymbol{K}| = \frac{2^{n/2}}{\sqrt{|\det \boldsymbol{H}|}}$$

The leading term of the asymptotic expansion is

$$F(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} \frac{e^{\lambda R(\boldsymbol{0})} g(\boldsymbol{0})}{\sqrt{|\det \boldsymbol{H}|}} \left(1 + o\left(\frac{1}{\lambda}\right)\right)$$

Example: Partition function of random matrix theory

A random matrix $\boldsymbol{M} = (a_{ij}) \in \mathbb{H}^{n\times n}$, the space of Hermite matrices with $\boldsymbol{M}^\dagger = \boldsymbol{M}$. We have

$$I(\lambda, \varepsilon) = \int_{\mathbb{H}^{n\times n}} \exp\left[\frac{\operatorname{tr} V(\boldsymbol{M})}{\varepsilon}\right] \mathrm{d}\boldsymbol{M}$$

The matrix includes diagonal elements $a_{11}, \cdots, a_{nn}$ and the off-diagonal elements $x_{ij} \pm iy_{ij}$. Therefore, we can choose these variables as the coordinates for $\mathbb{H}$ and obtain

$$\mathrm{d}\boldsymbol{M} = \left(\prod_i \mathrm{d}a_{ii}\right) \wedge \left(\prod_{i,j} \mathrm{d}x_{ij} \wedge \mathrm{d}y_{ij}\right)$$

The function V is chosen as

$$V(t, \boldsymbol{a}) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{2k} t^{2k}, \qquad \lambda_{2k} < 0$$

For a random matrix, the parameter a_0 is selected such that $I(\lambda, \varepsilon) = 1$. Since every Hermite matrix can be written as $\boldsymbol{M} = \boldsymbol{U}^\dagger \boldsymbol{\Lambda} \boldsymbol{U}$ with a unitary matrix $\boldsymbol{U}$. The unitary space is denoted as $\mathbb{U}$. The mapping $\mathbb{U}^{n\times n} \times \mathbb{R}^n \to \mathbb{H}^{n\times n}$, $(\boldsymbol{U}, \boldsymbol{\Lambda}) \mapsto \boldsymbol{U}^\dagger \boldsymbol{\Lambda} \boldsymbol{U} = \boldsymbol{M}$ is surjective. The integral becomes

$$I(\lambda, \varepsilon) = \frac{1}{N!} \int_{\mathbb{U}^{n\times n}} \mathrm{d}\boldsymbol{U} \int_{\mathbb{R}^n} \exp\left[\frac{\operatorname{tr} V(\boldsymbol{\Lambda})}{\varepsilon}\right] \cdot \prod_{i<j} \left(\lambda_i - \lambda_j\right)^2 \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n$$

The integral to study can be expressed as

$$F(\boldsymbol{a}, \varepsilon) = \int_{\mathbb{R}^n} \exp\left[\frac{1}{\varepsilon}\sum_{i=1}^{n} V(\lambda_i, \boldsymbol{a})\right] \cdot \prod_{i<j} \left(\lambda_i - \lambda_j\right)^2 \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n$$

- Exercise

Laplace's method 1

$$F(\lambda) = \int_0^{+\infty} e^{-\lambda\left(x + \frac{1}{x}\right)} \, \mathrm{d}x, \qquad \lambda \to +\infty$$

For this integral, we have $g(x) = 1$ and

$$R(x) = -\left(x + \frac{1}{x}\right), \qquad x_0 = 1, \qquad R_{\max} = R(x_0) = -2, \qquad R''(x_0) = -2$$

Now we have

$$F(\lambda) = e^{-2\lambda} \int_0^{+\infty} e^{-\lambda\left(x + \frac{1}{x} - 2\right)} \, \mathrm{d}x, \qquad \tilde{R}(x) = 2 - \left(x + \frac{1}{x}\right)$$

We want a coordinate transform $x = x(s)$ with $x(0) = 1$ such that

$$2 - \left(x + \frac{1}{x}\right) = -s^2, \qquad x(s) = \frac{s^2 + 2 + s\sqrt{s^2 + 4}}{2}$$

The Taylor series of $x(s)$ is

$$x(s) = 1 + s + \frac{s^2}{2} + \frac{s^3}{8} - \frac{s^5}{128} + \frac{s^7}{1024} + O(s^9)$$

$$x'(s) = 1 + s + \frac{3}{8}s^2 - \frac{5}{128}s^4 + \frac{7}{1024}s^6 + O(s^8)$$

Denote $\phi(s) = x'(s)$, the integral becomes

$$F(\lambda) \sim e^{-2\lambda} \int_{-\infty}^{+\infty} e^{-\lambda s^2} \phi(s)\, \mathrm{d}s = e^{-2\lambda} \sqrt{\frac{\pi}{\lambda}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n} n!} \frac{1}{\lambda^n}, \qquad \lambda \to +\infty$$

Based on the Taylor coefficients, we have

$$F(\lambda) \sim e^{-2\lambda} \sqrt{\frac{\pi}{\lambda}} \left(1 + \frac{3}{16}\lambda^{-1} - \frac{15}{512}\lambda^{-2} + \frac{105}{8192}\lambda^{-3} + O(\lambda^{-4})\right), \qquad \lambda \to +\infty$$

Laplace's method 2

$$F(x) = \int_0^{\frac{\pi}{4}} e^{x \cos t} \cos(nt)\, \mathrm{d}t\,, \qquad x \to +\infty, \qquad n \in \mathbb{Z}$$

For this integral, we have $g(t) = \cos nt$ and

$$R(t) = \cos t\,, \qquad t_0 = 0, \qquad R_{\max} = R(0) = 1, \qquad R'(0) = 0, \qquad R''(0) = -1$$

Now we have

$$F(x) \sim e^x \int_0^{\delta} e^{x(\cos t - 1)} \cos(nt)\, \mathrm{d}t\,, \qquad \bar{R}(t) = \cos t - 1$$

We want a coordinate transform $\tau = \tau(s)$ with $\tau(0) = 0$ such that

$$\cos\tau - 1 = -s^2, \qquad \tau(s) = \arccos(1 - s^2) = \sqrt{2}s + \frac{s^3}{6\sqrt{2}} + O(s^5)$$

Denote $\phi(s) = g(\tau(s))\, \tau'(s)$, the integral becomes

$$F(x) \sim e^x \int_0^{+\infty} e^{-\lambda s^2} \phi(s)\, \mathrm{d}s = \frac{e^x}{2} \sqrt{\frac{\pi}{x}} \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(0)}{2^{2n} n!} \frac{1}{x^n}, \qquad x \to +\infty$$

The Taylor series of $\phi(s)$ is obtained as

$$\phi(s) = \left(1 - n^2 s^2 + O(s^4)\right)\left(\sqrt{2} + \frac{s^2}{2\sqrt{2}} + O(s^4)\right) = \sqrt{2} + \frac{1 - 4n^2}{2\sqrt{2}} s^2 + O(s^4)$$

Based on the Taylor coefficients, we have

$$F(x) \sim e^x \sqrt{\frac{\pi}{2x}} \left(1 + \frac{1 - 4n^2}{8x} + O(x^{-2})\right), \qquad x \to +\infty$$

Laplace's method 3

$$F(x) = \int_0^1 e^{-x\sqrt{t}} \frac{\cos t}{\sqrt{t}}\, \mathrm{d}t\,, \qquad x \to +\infty$$

First we consider a change of variable $\tau = \sqrt{t}$, which gives

$$F(x) = 2\int_0^1 e^{-x\tau}\cos\tau^2\,\mathrm{d}\tau\,, \qquad x \to +\infty$$

Denote $\phi(\tau) = 2\cos\tau^2$ with its Taylor series

$$\phi(\tau) = 2 - \tau^4 + O(\tau^8)$$

The integral becomes

$$F(x) \sim \int_0^{+\infty} e^{-x\tau}\phi(\tau)\,\mathrm{d}\tau = \sum_{n=0}^{\infty}\frac{\phi^{(n)}(0)}{x^{n+1}} = \frac{2}{x} - \frac{24}{x^5} + O(x^{-9}), \qquad x \to +\infty$$

Laplace's method 4

$$F(x) = \int_0^1 e^t t^x (1+t^2)^{-x}\,\mathrm{d}t\,, \qquad x \to +\infty$$

First we rewrite the integrand as

$$F(x) = \int_0^1 e^{x\ln t - x\ln(1+t^2)} e^t\,\mathrm{d}t\,, \qquad x \to +\infty$$

Now we recognize $g(t) = e^t$ with

$$R(t) = \ln\left(\frac{t}{1+t^2}\right), \qquad t_0 = 1, \qquad R_{\max} = -\ln 2\,, \qquad R'(1) = 0, \qquad R''(1) = -1$$

The contribution comes from the right endpoint. The leading order term is

$$F(x) \sim \frac{e^{xR_{\max}}g(t_0)}{\sqrt{|R''(t_0)|}}\sqrt{\frac{\pi}{2x}}\left(1 + O(x^{-1})\right) = 2^{-x}e\sqrt{\frac{\pi}{2x}}\left(1 + O(x^{-1})\right), \qquad x \to +\infty$$

Laplace's method 5

$$F(x) = \int_0^{+\infty} e^{-x\cosh^2 t}\,\mathrm{d}t\,, \qquad x \to +\infty$$

With a change of variable $\tau = \sinh t$, we have

$$F(x) = e^{-x}\int_0^{+\infty} e^{-x\tau^2}\frac{1}{\sqrt{1+\tau^2}}\,\mathrm{d}\tau\,, \qquad x \to +\infty$$

Based on the following Taylor series

$$\phi(\tau) = \frac{1}{\sqrt{1+\tau^2}} = \sum_{n=0}^{\infty}\binom{-1/2}{n}\tau^{2n} = 1 - \frac{1}{2}\tau^2 + O(\tau^4)$$

$$F(x) \sim e^{-x}\int_0^{+\infty} e^{-x\tau^2}\phi(\tau)\,\mathrm{d}\tau = \frac{e^{-x}}{2}\sqrt{\frac{\pi}{x}}\sum_{n=0}^{\infty}\frac{\phi^{(2n)}(0)}{2^{2n}n!}\frac{1}{x^n}, \qquad x \to +\infty$$

The integral becomes

$$F(x) \sim \frac{e^{-x}}{2}\sqrt{\frac{\pi}{x}}\left(1-\frac{1}{4x}+O(x^{-2})\right), \qquad x \to +\infty$$

Laplace's method 6

$$F(x) = \int_{-\infty}^{+\infty} e^{x(t-e^t)}\,\mathrm{d}t, \qquad x \to +\infty$$

With $R(t) = t - e^t$, we can recognize

$$t_0 = 0, \qquad R_{\max} = -1, \qquad \bar{R}(t) = t - e^t + 1, \qquad \bar{R}'(0) = 0$$

We want a coordinate transform $\tau = \tau(s)$ with $\tau(0) = 0$ such that

$$\tau - e^\tau + 1 = -s^2, \qquad \tau(s) = \sum_{n\geq 1} a_n s^n = \sqrt{2}s - \frac{1}{3}s^2 + \frac{\sqrt{2}}{18}s^3 + O(s^4)$$

Denote $\phi(s) = \tau'(s)$, the integral becomes

$$\phi(s) = \sqrt{2} - \frac{2}{3}s + \frac{\sqrt{2}}{6}s^2 + O(s^3)$$

$$F(x) \sim e^{-x}\int_{-\infty}^{+\infty} e^{-xs^2}\phi(s)\,\mathrm{d}s = e^{-x}\sqrt{\frac{\pi}{x}}\sum_{n=0}^{\infty}\frac{\phi^{(2n)}(0)}{2^{2n}n!}\frac{1}{x^n}, \qquad x \to +\infty$$

Based on the Taylor coefficients, we have

$$F(x) \sim e^{-x}\sqrt{\frac{2\pi}{x}}\left(1+\frac{1}{12}x^{-1}+O(x^{-2})\right), \qquad x \to +\infty$$

Laplace's method 7

We want to prove the following asymptotic relation

$$\sum_{s=0}^{n}\binom{n}{s}\frac{s!}{n^s} \sim \sqrt{\frac{\pi n}{2}}, \qquad n \to +\infty$$

The result of the integral below is given

$$\int_0^{+\infty} e^{-nt}t^s\,\mathrm{d}t = \frac{s!}{n^{s+1}}, \qquad s \in \mathbb{N}$$

From the binomial theorem, we have

$$\int_0^{+\infty} e^{-nt}(1+t)^n\,\mathrm{d}t = \sum_{s=0}^{n}\binom{n}{s}\int_0^{+\infty} e^{-nt}t^s\,\mathrm{d}t = \sum_{s=0}^{n}\binom{n}{s}\frac{s!}{n^{s+1}}$$

Therefore, we need to study the asymptotic behavior of the following integral

$$F(n) = \int_0^{+\infty} e^{-nt}(1+t)^n \,\mathrm{d}t = \int_0^{+\infty} e^{-nt+n\ln(1+t)} \,\mathrm{d}t\,, \qquad n \to +\infty$$

With $R(t) = -t + \ln(1+t)$, we can recognize

$$t_0 = 0, \qquad R_{\max} = 0, \qquad R'(0) = 0, \qquad R''(0) = -1$$

The leading order term is thus obtained as

$$F(n) \sim \frac{e^{xR_{\max}} g(t_0)}{\sqrt{|R''(t_0)|}} \sqrt{\frac{\pi}{2x}} \big(1 + O(x^{-1})\big) = \sqrt{\frac{\pi}{2n}} \big(1 + O(n^{-1})\big), \qquad n \to +\infty$$

Finally, we have

$$\sum_{s=0}^{n} \binom{n}{s} \frac{s!}{n^s} = nF(n) \sim \sqrt{\frac{\pi n}{2}} \big(1 + O(n^{-1})\big), \qquad n \to +\infty$$

Laplace's method 8

$$F(x) = \int_0^{+\infty} e^{-xt} \sin t \,\mathrm{d}t\,, \qquad x \to +\infty$$

We directly recognize $\phi(t) = \sin t$ with its Taylor series

$$\phi(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

With the dominant contribution from the left endpoint $t_0 = 0$, we have

$$F(x) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{x^{n+1}} = \sum_{n=0}^{\infty} \frac{\phi^{(2n+1)}(0)}{x^{2n+2}} = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{2n}} = \frac{1}{x^2+1}$$

Asymptotic Analysis of Integrals (3): Steepest Descents

We analyze the asymptotic expansion of the following integral

$$F(\lambda) = \int_\gamma e^{\lambda h(z)} g(z)\, \mathrm{d}z\,, \qquad \lambda \to \infty$$

With a curve $\gamma \in \Omega \subseteq \mathbb{C}$ and holomorphic functions $h(z)$ and $g(z)$ in $\mathbb{O}(\Omega)$. If another curve $\tilde{\gamma}$ is homotopic to γ, then the result is the same. The goal is to find an appropriate $\tilde{\gamma}$ such that we can perform asymptotic analysis of the integral.

- Contour lines of analytic functions (4.2)

Consider an analytic function $h \in \mathbb{O}(\Omega)$, we have

$$h(z) = u(x,y) + iv(x,y), \qquad z = x + iy$$

The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x$$

Both $u(x,y)$ and $v(x,y)$ are harmonic

$$u_{xx} = v_{xy} = -u_{yy}, \qquad v_{xx} = -u_{xy} = -v_{yy}$$

The derivative $h'(z)$ can be expressed as

$$h'(z) = u_x + iv_x = u_x - iu_y = v_y + iv_x$$

For $z_0 = x_0 + iy_0$, denote $h(z_0) = u_0 + iv_0$. Define two sets as

$$\gamma_u = \{z \in \Omega \mid u(x,y) = u_0\}, \qquad \gamma_v = \{z \in \Omega \mid v(x,y) = v_0\}$$

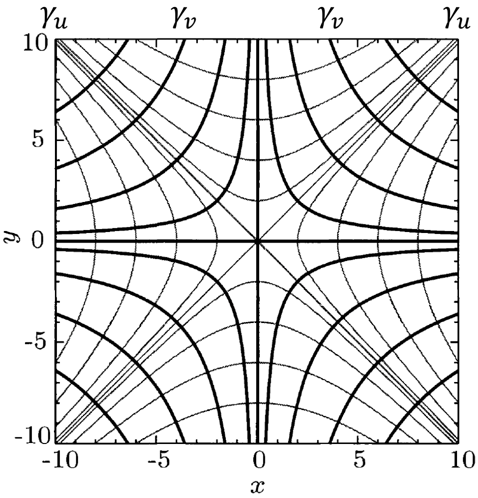
If $h'(z_0) \neq 0$, we have

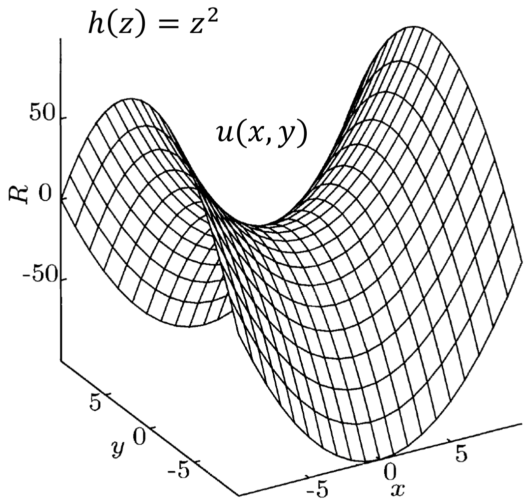
$$\nabla u(x_0, y_0) \neq 0, \qquad \nabla v(x_0, y_0) \neq 0$$

Without loss of generality, consider $u_x \neq 0$. The implicit function theorem implies $x = \phi(y)$. Also note that

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0$$

This shows that γ_u and γ_v are orthogonal to each other.





Example

$$h(z) = z^2, \qquad u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy$$

γ_u are plotted by light curves, while γ_v are plotted by dark curves. They are defined by

$$\gamma_u = \{x^2 - y^2 = c\}, \qquad \gamma_v = \{2xy = c\}$$

Saddle points (4.4)

If $h'(z_0) = 0$ and $h''(z_0) \neq 0$, then we have

$$h(z) = h(z_0) + \frac{1}{2}h''(z_0)(z - z_0)^2 + o((z - z_0)^2)$$

There exists a transformation $z = z(w)$ such that $h(w) = h(z_0) + w^2$. This goes back to the analysis of function $h(z) = z^2$, which has a critical point $z_0 = 0$. For a general critical point z_0, the derivatives satisfy

$$u_x(x_0, y_0) = 0, \qquad u_y(x_0, y_0) = 0, \qquad \mathrm{Hess}(u) = \begin{vmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{vmatrix} = -u_{xx}^2 - u_{xy}^2 < 0$$

This shows that (x_0, y_0) is a saddle point of $u(x,y)$. Similar argument for $v(x,y)$.

If $h'(z_0) = h''(z_0) = \cdots = h^{(k-1)}(z_0) = 0$ and $h^{(k)}(z_0) \neq 0$, then we have

$$h(z) = h(z_0) + \frac{1}{k!}h^{(k)}(z_0)(z - z_0)^k + o((z - z_0)^k)$$

We then can obtain a transformation that leads to $h(w) = h(z_0) + w^k$.

- Steepest descents (4.3, 4.5)

The path $\tilde{\gamma} = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ satisfies that each γ_j is a portion of contour levels γ_u or γ_v. For γ_i that is along γ_u, we have

$$\left| \int_{\gamma_i} e^{\lambda h(z)} g(z)\, \mathrm{dz} \right| = \left| e^{\lambda u} \int_{\gamma_i} e^{i\lambda v} g(z)\, \mathrm{dz} \right| \leq e^{\lambda u} \int_{\gamma_i} |g(z)|\, \mathrm{dz} = Ce^{\lambda u}$$

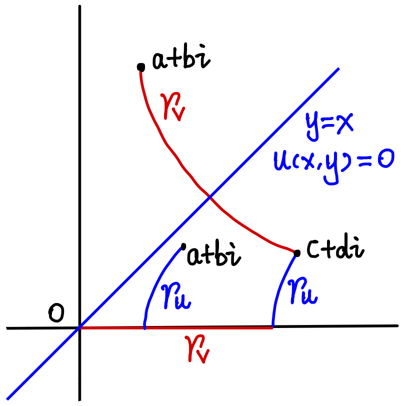
We choose γ_i such that $u < 0$, and the contribution goes to 0 as $\lambda \to \infty$. On the other hand, for γ_j that is along γ_v, we have

$$\int_{\gamma_j} e^{\lambda h(z)} g(z)\, \mathrm{dz} = e^{i\lambda v} \int_{\gamma_j} e^{\lambda u} g(z)\, \mathrm{dz}$$

$$= e^{i\lambda v} \int_{\alpha_j}^{\beta_j} e^{\lambda u(x(t),y(t))} g\big(x(t), y(t)\big)[x'(t) + iy'(t)]\, \mathrm{dt}$$

The integral then can be analyzed by Laplace's method. Typically, the first portion is along γ_v in order to reach γ_u with $u < 0$.

Example 1: Contribution from endpoints



$$F(\lambda) = \int_0^{a+bi} e^{-\lambda z^2} \mathrm{d}z\,, \qquad a, b > 0$$

We have $h(z) = -z^2$ with

$$u = y^2 - x^2, \qquad v = -2xy$$

Start from $z = 0$, we go along γ_v: $0 \to a_1$ to reduce u. Based on different locations $a + bi$, we need to construct different paths $\tilde{\gamma}$, and there are two cases to consider:

- Case 1: $a > b$. We can easily find one γ_u that connects $z = a + bi$ to the real axis.

$$\tilde{\gamma}: (\gamma_v: 0 \to a_1) \cup (\gamma_u: a_1 \to a + bi)$$

We only need to evaluate along the first segment $\gamma_v: 0 \to a_1$, which gives

$$F(\lambda) \sim \int_0^{a_1} e^{-\lambda x^2} \mathrm{d}x \sim \int_0^{\infty} e^{-\lambda x^2} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\left(1 + O\left(\frac{1}{\lambda}\right)\right), \qquad \lambda \to \infty$$

- Case 2: $a < b$. We still need another $\gamma_v: c + di \to a + bi$ to connect.

$$\tilde{\gamma}: (\gamma_v: 0 \to a_1) \cup (\gamma_u: a_1 \to c + di) \cup (\gamma_v: c + di \to a + bi)$$

The first segment gives the same result. On the third segment, we have

$$xy = ab = cd, \qquad x = \frac{ab}{y}, \qquad z = \frac{ab}{y} + iy$$

The integral becomes

$$I_3(\lambda) = e^{-2i\lambda ab} \int_d^b e^{\lambda\left(y^2 - \frac{a^2 b^2}{y^2}\right)} \left(-\frac{ab}{y^2} + i\right) \mathrm{d}y$$

Using Laplace's method, we have

$$R_{\max} = R(b) = b^2 - a^2, \qquad g(b) = -\frac{a}{b} + i, \qquad R'(b) = \frac{2(a^2 + b^2)}{b}$$

The asymptotic expansion is

$$F(\lambda) \sim e^{-2i\lambda ab} e^{\lambda R_{\max}} \left(\frac{g(b)}{R'(b)}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right)\right) = -\frac{e^{-\lambda(a+bi)^2}}{2(a + bi)}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty$$

The contribution from $I_3(\lambda)$ is of much greater order than $I_1(\lambda)$.

Example 2: Contribution from saddle points (4.5)

In practice, many exponential integrals are specified on contours C that have no endpoints, either because the contour is closed or because the contour tends to infinity in both directions. Along integration path $\gamma_v(t)$, we usually have $u(x(t), y(t))$ taking its maximum at the saddle point of $h(z)$, where the dominant contribution to the integral arises.

If $h \in \Omega$ only has one non-degenerate critical point z_0 with $h'(z_0) = 0$ and $h''(z_0) \neq 0$, then it is a saddle point. For a curve $\gamma_v(t)$ that goes through this point z_0, denote

$$\gamma_v(t) = \big(x(t), y(t)\big), \qquad t \in [\alpha, \beta], \qquad \gamma_v(t_0) = z_0$$

The asymptotic expansion of the integral $F(\lambda)$ becomes

$$F(\lambda) \sim \int_\alpha^\beta e^{\lambda h(z(t))} g\big(z(t)\big)\, z'(t)\, \mathrm{d}t = e^{\lambda h(z_0)} \int_\alpha^\beta e^{\lambda \tilde{h}(t)} g\big(z(t)\big)\, z'(t)\, \mathrm{d}t$$

Note that $\tilde{h}(t) = h(z(t)) - h(z_0)$ is real as on $\gamma_v(t)$. From Laplace's method, we have

$$F(\lambda) \sim e^{\lambda h(z_0)} \left[\sqrt{\frac{2\pi}{|\tilde{h}''(t_0)|}}\, g(z_0)\, z'(t_0) \frac{1}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right) \right], \qquad \lambda \to \infty$$

To remove the dependence on parametrization, from $h'(z_0) = 0$ we have

$$\tilde{h}''(t_0) = h''(z_0)[z'(t_0)]^2$$

Finally, we obtain the expression

$$F(\lambda) \sim e^{\lambda h(z_0)} \left[e^{i \arg z'(t_0)} \sqrt{\frac{2\pi}{|h''(z_0)|}}\, g(z_0) \frac{1}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right) \right], \qquad \lambda \to \infty$$

The angle of the tangent at which γ_v passes through the saddle point z_0 is needed.

- Airy function (4.7)

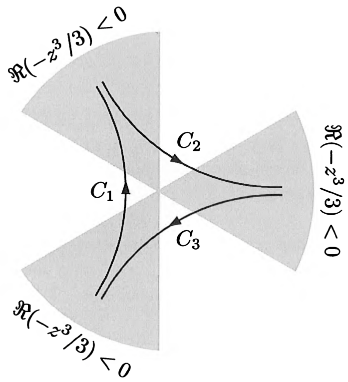
The Airy function is the solution of Airy's equation

$$y''(x) - xy(x) = 0$$

Using the form of a Fourier-Laplace integral, we can obtain

$$y_j(x) = \frac{1}{2\pi i} \int_{C_j} e^{zx - z^3/3}\, \mathrm{d}z$$

They can be shown to satisfy Airy's equation. Note that



$$y_1(x) + y_2(x) + y_3(x) = 0$$

The function $y_1(x)$ is denoted as $\mathrm{Ai}(x)$. Denote $x = re^{i\theta}$, and for a fixed θ we aim to study the asymptotic behavior as $r \to \infty$. The integral becomes

$$y_1\big(re^{i\theta}\big) = \frac{1}{2\pi i} \int_{C_1} e^{re^{i\theta} z - z^3/3}\, \mathrm{d}z$$

First, we find a scaling $z = \rho w$ such that the two terms having the same order of magnitude.

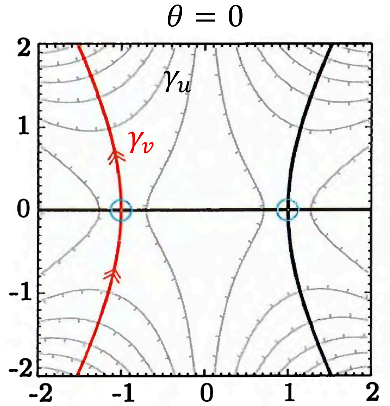
$$r\rho w \sim \rho^3 w^3, \qquad \rho \propto \sqrt{r}$$

Now denote $\gamma_1 = C_1/\sqrt{r}$ and $\lambda = r^{3/2}$, still using z as the integration variable, we have

$$y_1\big(re^{i\theta}\big) = \frac{\sqrt{r}}{2\pi i} \int_{\gamma_1} e^{\lambda h_\theta(z)}\, \mathrm{d}z\,, \qquad h_\theta(z) = e^{i\theta} z - \frac{z^3}{3}$$

- When $\theta = 0$, we have $x \to +\infty$. The critical points are obtained as

$$h_0(z) = z - \frac{z^3}{3}, \qquad z_1 = 1, \qquad z_2 = -1$$



With $z = x + iy$, the real and imaginary parts are

$$u(x,y) = x\left(1 - \frac{1}{3}x^2 + y^2\right)$$

$$v(x,y) = y\left(1 - x^2 + \frac{1}{3}y^2\right)$$

At the critical points, we have

$$u_1 = \frac{2}{3}, \qquad u_2 = -\frac{2}{3}, \qquad v_1 = v_2 = 0$$

This implies that z_1 and z_2 lie on one of the γ_v: $\{v = 0\}$ curves, which is the real axis. For the path C_1, we can directly deform it to one hyperbola γ_v: $\{v = 0\}$ shown in red. Given that

$$h_0(z_2) = -\frac{2}{3}, \qquad h''(z_2) = 2, \qquad \theta(z_2) = \arg z'(t_0) = \frac{\pi}{2}$$

Then we have

$$F_0(\lambda) = \int_{\gamma_v} e^{\lambda h_0(z)}\,\mathrm{d}z = ie^{-2\lambda/3}\sqrt{\frac{\pi}{\lambda}}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \qquad x \to +\infty$$

With $\lambda = x^{3/2}$, the asymptotic expansion becomes

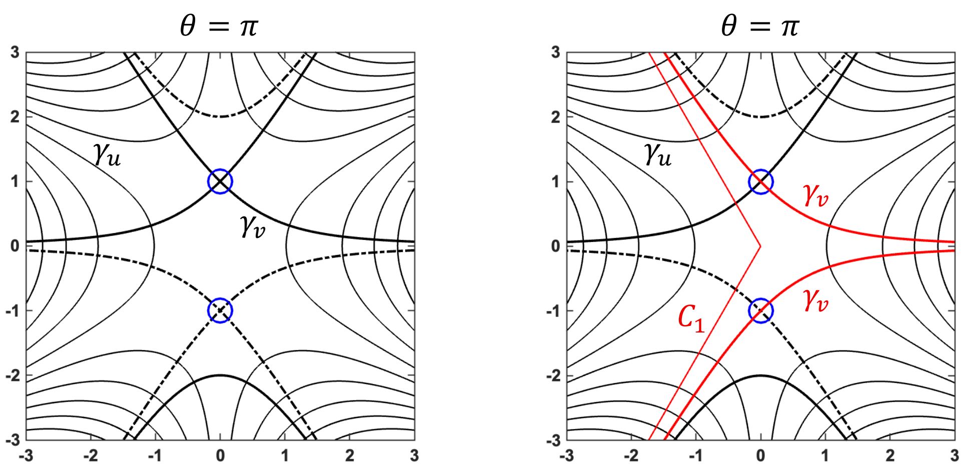
$$y_1(x) = \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}\,x^{1/4}}\left[1 + O\left(\frac{1}{x^{3/2}}\right)\right], \qquad x \to +\infty$$

- When $\theta = \pi$, we have $x \to -\infty$. The critical points are obtained as

$$h_\pi(z) = -z - \frac{z^3}{3}, \qquad z_1 = i, \qquad z_2 = -i$$

With $z = x + iy$, the real and imaginary parts are

$$u(x,y) = -x\left(1 + \frac{1}{3}x^2 - y^2\right), \qquad v(x,y) = -y\left(1 + x^2 - \frac{1}{3}y^2\right)$$



At the critical points, we have

$$u_1 = u_2 = 0, \qquad v_1 = -\frac{2}{3}, \qquad v_2 = \frac{2}{3}$$

Start in the lower half-plane, to obtain $u < 0$ we need to follow the curve $\gamma_v\colon \{v = v_2\}$ which goes through the saddle point z_2. However, it deviates from our original C_1. We need to connect at infinity the curve $\gamma_v\colon \{v = v_1\}$ and close the loop. Given that

$$h_0(z_1) = -\frac{2}{3}i, \qquad h''(z_1) = -2i, \qquad \theta(z_1) = \arg z'(t_0) = \frac{3\pi}{4}$$

$$h_0(z_2) = \frac{2}{3}i, \qquad h''(z_2) = 2i, \qquad \theta(z_2) = \arg z'(t_0) = \frac{\pi}{4}$$

Then we have

$$F_\pi(\lambda) = -e^{-\frac{2i\lambda}{3}} e^{-\frac{i\pi}{4}} \sqrt{\frac{\pi}{\lambda}} \left[1 + O\left(\frac{1}{\lambda}\right)\right] + e^{\frac{2i\lambda}{3}} e^{\frac{i\pi}{4}} \sqrt{\frac{\pi}{\lambda}} \left[1 + O\left(\frac{1}{\lambda}\right)\right], \qquad x \to -\infty$$

With $\lambda = |x|^{3/2}$, the asymptotic expansion becomes

$$y_1(x) = \frac{1}{\sqrt{\pi}\,|x|^{1/4}} \sin\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)\left[1 + O\left(\frac{1}{|x|^{3/2}}\right)\right], \qquad x \to -\infty$$

We notice that completely different asymptotic formula can hold for representations of entire functions as $x \to \infty$ in different sectors. This is called the Stokes phenomenon.

➢ Effect of branch points (4.8)

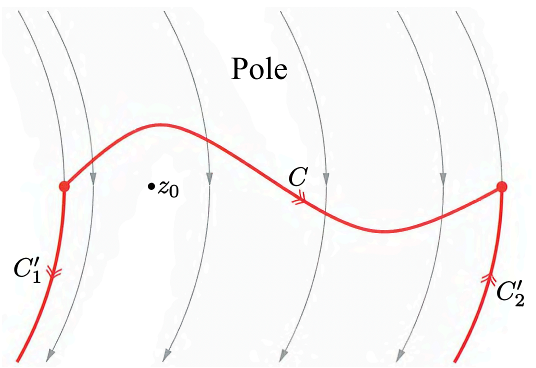
$$F(\lambda) = \int_\gamma e^{\lambda h(z)} g(z)\, \mathrm{d}z\,, \qquad \lambda \to \infty$$

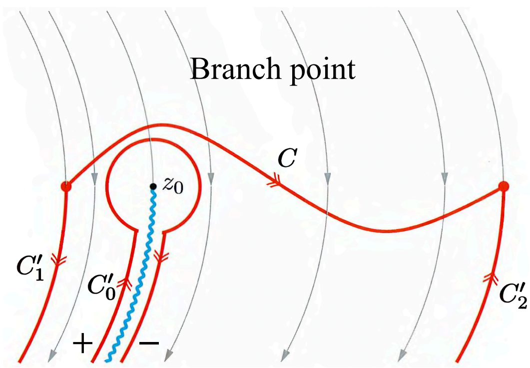
We consider $h \in \mathbb{O}(\Omega)$ and $g(z)$ is a meromorphic function or a multivalued function. The curve $\gamma \in \Omega$ does not intersect with the branch cuts. When $g(z) \in m(\Omega)$, we need to consider the additional contributions from the residues, besides those from the steepest descent paths.

$$\int_\gamma = \int_{\tilde{\gamma}} + 2\pi i \sum_{j=1}^{N} \mathrm{Res}\left(e^{\lambda h(z)} g(z), z_j\right), \qquad \tilde{\gamma} = C_1' \cup C_2'$$

The asymptotic analysis of the residue at z_0 is as follows. Given the Laurent series

$$g(z) = \sum_{k \geq -m} g_k (z - z_0)^k\,, \qquad h(z) = \sum_{k \geq 0} h_k (z - z_0)^k$$





The residue contribution is a factor $e^{\lambda h(z_0)}$ times a polynomial $p(\lambda)$ of degree $m-1$.

$$\mathrm{Res}\left(e^{\lambda h(z)}g(z), z_0\right) = e^{\lambda h(z_0)} \cdot p_{m-1}(\lambda)$$

When $g(z)$ is a multivalued function, we start with the following example.

$$I = \oint_{|z|=\rho} f(z)\,\mathrm{d}z\,, \qquad f(z) = z^\sigma, \qquad \sigma \notin \mathbb{Z}$$

The branch cut is the positive real axis, and we choose the branch $\theta \in [0,2\pi)$. Directly using the anti-derivative $F(z)$, we have

$$F(z) = \frac{z^{\sigma+1}}{\sigma+1}, \qquad I = F\left(\rho e^{i2\pi}\right) - F(\rho) = \frac{\rho^{\sigma+1}}{\sigma+1}\left(e^{i2\pi\sigma} - 1\right)$$

Note that the result depends on ρ. If $\sigma + 1 > 0$, then we have $I \to 0$ as $\rho \to 0$.

To apply the steepest descent method for a multivalued function $g(z)$, we first need to choose a new branch cut that follows γ_v in the direction of decreasing u. Assume $g(z)$ is given as

$$g(z) = (z-z_0)^\sigma \tilde{g}(z), \qquad \tilde{g}(z) \in \mathbb{O}(\Omega), \qquad g'(z_0) \neq 0$$

The same can be applied if $g(z) = \ln(z-z_0)\,\tilde{g}(z)$.

- When $\sigma > -1$, the contribution from the circle $|z-z_0| = \rho$ goes to zero as $\rho \to 0$.
- When $\sigma < -1$, using integration by parts we have

$$\int_{C_0'} e^{\lambda h(z)}\tilde{g}(z)(z-z_0)^\sigma\,\mathrm{d}z = e^{\lambda h(z)}\tilde{g}(z)\frac{(z-z_0)^{\sigma+1}}{\sigma+1}\Bigg|_{\partial C_0'} - \int_{C_0'}\left[e^{\lambda h(z)}\tilde{g}(z)\right]'\frac{(z-z_0)^{\sigma+1}}{\sigma+1}\mathrm{d}z$$

The boundary terms is neglected as the branch cut is in the direction of $u < 0$, with small $e^{\lambda h(z)}$ at the endpoints as $\lambda \to \infty$. The new integral is of order $\sigma + 1$, and by repeatedly doing this we can obtain a final integral with $\sigma > -1$.

Now for $\sigma > -1$, we study the contribution along the branch cut. Denote $C_0' = \gamma_0^+ \cup (\gamma_0^-)^-$, with both γ_0^+ and γ_0^- pointing towards z_0. Along the branch cut, we have

$$(z-z_0)_-^\sigma = e^{i2\pi\sigma}(z-z_0)_+^\sigma$$

The integral along C_0' becomes

$$F_0(\lambda) = \int_{\gamma_0^+} - \int_{\gamma_0^-} = \left(1 - e^{i2\pi\sigma}\right)\int_{\gamma_0^+} e^{\lambda h(z)}\tilde{g}(z)(z-z_0)_+^\sigma\,\mathrm{d}z$$

Then we apply Laplace's method (endpoint) with contribution from the branch point $z = z_0$.

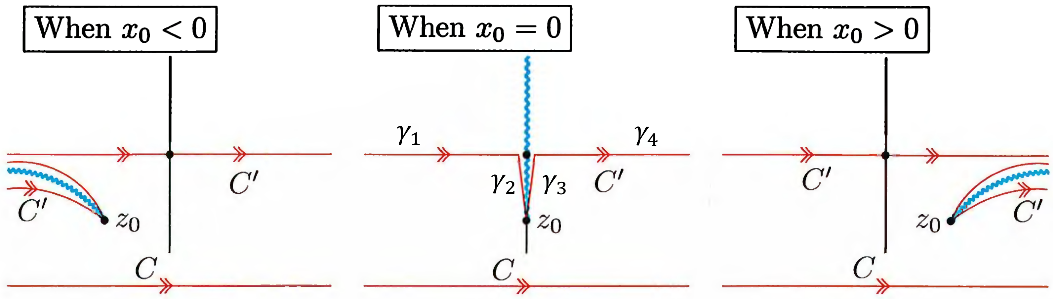
Example

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{e^{-\lambda x^2}\cos(2\lambda\beta x)}{\sqrt{x^2+1}}\,\mathrm{d}x\,, \qquad \beta > 1, \qquad \lambda \to +\infty$$

The effect of branch point is shown below, with $g(z)$ and its branch point z_0 given as

$$g(z) = \frac{1}{\sqrt{(z-x_0)^2+1}}, \qquad z_0 = x_0 + i$$

In our example we have $x_0 = 0$, and since $\beta > 1$ the saddle point is above the branch point.



Due to symmetry, the integral is equivalent to

$$F(\lambda) = \int_{\mathbb{R}} \frac{e^{-\lambda z^2 + 2i\lambda\beta z}}{\sqrt{z^2+1}}\,\mathrm{d}z = e^{-\lambda\beta^2}\int_{\mathbb{R}} e^{\lambda h(z)} g(z)\,\mathrm{d}z\,, \qquad h(z) = -(z-i\beta)^2$$

The steepest descent path is constructed as

$$F_0(\lambda) = \int_\gamma e^{\lambda h(z)} g(z)\,\mathrm{d}z\,, \qquad \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$$

In the branch $(z^2+1)^{1/2} > 0$ for $z \in \mathbb{R}$, we have

$$\mathrm{Im}\,\sqrt{z^2+1} < 0, \qquad z \in \gamma_1 \cup \gamma_2, \qquad \mathrm{Im}\,\sqrt{z^2+1} > 0, \qquad z \in \gamma_3 \cup \gamma_4$$

On γ_1 and γ_4, their contributions cancel out for the leading order term, because $g(z)$ results in opposite signs on different sides of the branch cut at the saddle point $z = i\beta$ (i.e., $t = 0$).

$$\int_{\gamma_1} e^{\lambda h(z)} g(z)\,\mathrm{d}z = \int_{-\infty}^{0} \frac{e^{-\lambda t^2}}{\sqrt{(t+i\beta)^2+1}}\,\mathrm{d}t = \int_0^{+\infty} \frac{e^{-\lambda t^2}}{\sqrt{(t-i\beta)^2+1}}\,\mathrm{d}t$$

$$\int_{\gamma_4} e^{\lambda h(z)} g(z)\,\mathrm{d}z = \int_0^{+\infty} \frac{e^{-\lambda t^2}}{\sqrt{(t+i\beta)^2+1}}\,\mathrm{d}t$$

Along the branch cut, we have

$$\int_{\gamma_2} e^{\lambda h(z)} g(z)\,\mathrm{d}z = \int_\beta^1 \frac{e^{\lambda(\beta-t)^2}}{\sqrt{1-t^2}}\,i\mathrm{d}t = \int_1^\beta \frac{e^{\lambda(\beta-t)^2}}{\sqrt{t^2-1}}\,\mathrm{d}t\,, \qquad \sqrt{1-t^2} = -i\sqrt{t^2-1}$$

$$\int_{\gamma_3} e^{\lambda h(z)} g(z)\,\mathrm{d}z = \int_1^\beta \frac{e^{\lambda(\beta-t)^2}}{\sqrt{1-t^2}}\,i\mathrm{d}t = \int_1^\beta \frac{e^{\lambda(\beta-t)^2}}{\sqrt{t^2-1}}\,\mathrm{d}t\,, \qquad \sqrt{1-t^2} = i\sqrt{t^2-1}$$

Therefore, we have

$$F_0(\lambda) \sim 2\int_1^\beta \frac{e^{\lambda(\beta-t)^2}}{\sqrt{t^2-1}}\,\mathrm{d}t\,, \qquad \tilde{h}(t) = (\beta-t)^2, \qquad \lambda \to \infty$$

The dominant contribution arises at the branch point $z = z_0 = i$ (i.e., $t = 1$). To evaluate this integral, consider a change of variable

$$s = \tilde{h}_{\max} - \tilde{h}(t) = (\beta-1)^2 - (\beta-t)^2, \qquad t(s=0) = 1$$

This leads to the transform

$$t = \beta - \sqrt{(\beta-1)^2 - s}$$

The integral becomes

$$F_0(\lambda) \sim 2e^{\lambda(\beta-1)^2}\int_0^{(\beta-1)^2} \frac{e^{-\lambda s}}{\sqrt{t(s)^2-1}}\,t'(s)\,\mathrm{d}s$$

From the power series expansion

$$t(s) = 1 + \frac{s}{2(\beta-1)} + \frac{s^2}{8(\beta-1)^3} + o(s^2)$$

We have

$$\frac{t'(s)}{\sqrt{t(s)^2-1}} = \frac{1+O(s)}{2\sqrt{(\beta-1)s}}, \qquad F_0(\lambda) \sim \frac{e^{\lambda(\beta-1)^2}}{\sqrt{\beta-1}}\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}}\big(1+O(s)\big)\,\mathrm{d}s$$

From Watson's Lemma, we have

$$F_0(\lambda) \sim \frac{e^{\lambda(\beta-1)^2}}{\sqrt{\beta-1}}\sqrt{\frac{\pi}{\lambda}}\left(1+O\left(\frac{1}{\lambda}\right)\right), \qquad \lambda \to \infty$$

The leading order term of the asymptotic expansion is finally obtained as

$$F(\lambda) = e^{-\lambda\beta^2}F_0(\lambda) \sim \frac{e^{-\lambda(2\beta-1)}}{\sqrt{\beta-1}}\sqrt{\frac{\pi}{\lambda}}\left(1+O\left(\frac{1}{\lambda}\right)\right), \qquad \lambda \to +\infty$$

- Asymptotic analysis of integral transformation (4.8.1)

1. **Fourier transform**: For $f \in L^2(\mathbb{R},\mathbb{C})$, we have $\hat{f} \in L^2(\mathbb{R},\mathbb{C})$ given as

$$\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-ikx}\,\mathrm{d}x\,, \qquad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k)e^{ikx}\,\mathrm{d}k$$

2. **Laplace transform**: For $f: \mathbb{R}_{>0} \to \mathbb{C}$ with $c \in \mathbb{R}$ such that $f(t) = O(e^{ct})$ as $t \to \infty$, we have a holomorphic function $F \in \mathbb{O}(\Omega_c)$, $\Omega_c = \{s \in \mathbb{C} \mid \operatorname{Re} s > c\}$ given as

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t\,, \qquad f(t) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} F(s)e^{st}\,\mathrm{d}s\,, \qquad c' > c$$

3. **Mellin transform**: For $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ satisfying

$$f(x) = \begin{cases} O(x^{-a}), & x \to 0^+ \\ O(x^{-b}), & x \to +\infty \end{cases}$$

Define a domain $\Omega_{ab} = \{s \in \mathbb{C} \mid a < \operatorname{Re} s < b\}$, then we have $\varphi \in \mathbb{O}(\Omega_{ab})$ given as

$$\varphi(s) = \int_0^\infty f(x) x^{s-1}\,\mathrm{d}t\,, \qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi(s) x^{-s}\,\mathrm{d}s\,, \qquad c \in (a,b)$$

4. **Z transform**: For $\{a_n\}\colon \mathbb{Z} \to \mathbb{C}$, we have $f(z) \in \mathbb{O}(\Omega)$ for a domain Ω given as

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}\,, \qquad a_n = \frac{1}{2\pi i}\oint_\gamma f(z) z^{n-1}\,\mathrm{d}z$$

An equivalent description is to take the coefficient of the generating function $f(z)$

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^{n}\,, \qquad a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}\,\mathrm{d}z$$

Example 1: Mellin transform & Gamma function

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s\,\mathrm{d}x$$

We can equivalently write

$$\Gamma(s+1) = \int_0^\infty e^{h(x;s)}\,\mathrm{d}x\,, \qquad h(x;s) = -x + s\ln x$$

The saddle point is $x_0 = s$ which depends on s. Previously we let $x = sy$ and obtain

$$\Gamma(s+1) = s^{s+1}\int_0^\infty e^{s(\ln y - y)}\,\mathrm{d}y$$

Example 2: Z transform & Stirling formula

$$e^z = \sum_{n\ge 0} \frac{z^n}{n!}, \qquad \frac{1}{n!} = \frac{1}{2\pi i}\oint_\gamma \frac{e^z}{z^{n+1}}\,\mathrm{d}z$$

The contour is constructed as $\gamma\colon z = \rho e^{i\theta}$ with $\theta \in [-\pi,\pi)$. We want to properly choose ρ to perform asymptotic analysis. On this contour, we have

$$I = \frac{1}{2\pi i}\oint_\gamma \frac{e^z}{z^{n+1}}\,\mathrm{d}z = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{h(\theta;n,\rho)}\,\mathrm{d}\theta\,, \qquad h(\theta;n,\rho) = \rho e^{i\theta} - n\ln\rho - in\theta$$

The saddle point $\theta = \theta_0$ satisfies $\rho e^{i\theta} = n$, which gives $\rho = n$ and $\theta = 0$. With the change of variable $\rho = n$, we have

$$h = ne^{i\theta} - n\ln n - in\theta, \qquad \frac{1}{n!} = \frac{n^{-n}}{2\pi}\int_{-\pi}^{\pi} e^{n(e^{i\theta} - i\theta)}\,\mathrm{d}\theta$$

Example 3: Fourier transform

Consider $f(x) = O(e^{-c|x|})$ when $x \to \pm\infty$ for $c > 0$. The Taylor coefficients are bounded as

$$|a_n| = \left|\frac{\hat{f}^{(n)}(k)}{n!}\right| = \frac{1}{n!}\left|\int_{\mathbb{R}} (-ix)^n f(x) e^{-ikx}\, \mathrm{d}x\right| \le \frac{1}{n!}\left|\int_{\mathbb{R}} A e^{-c|x|}|x|^n\, \mathrm{d}x\right| = \frac{2A}{c^{n+1}}$$

The radius of convergence is

$$R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} \ge c$$

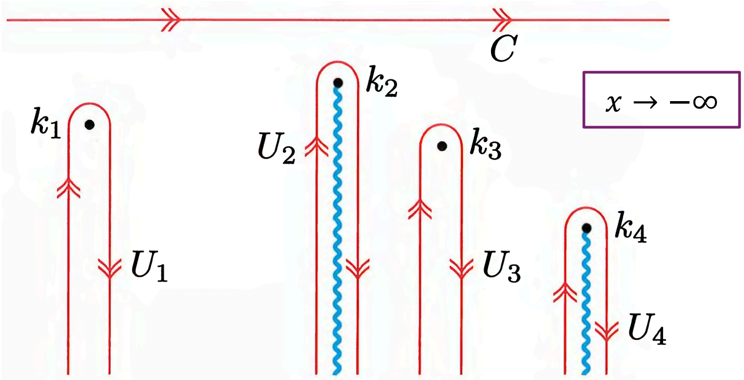
So $\hat{f}(k)$ is analytic in $\{k \in \mathbb{C} \mid |\mathrm{Im}\, k| \le c\}$ near the real axis. The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k) e^{ikx}\, \mathrm{d}k$$

When $x \to -\infty$, denote $x = -\lambda$ and $k = p + iq$, we have

$$h(k) = -ik, \qquad u = q, \qquad v = -p$$

The steepest descent paths γ_v are straight vertical lines pointing downward. For $x \to +\infty$, the paths will be pointing upward.



Example 4: Laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} F(s) e^{st}\, \mathrm{d}s\,, \qquad t \to \infty$$

When $t \to \infty$, we have $h(s) = s$ and the steepest descent paths γ_v are straight horizontal lines pointing to the left.

Example 5: Mellin transform for summation

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi(s) x^{-s}\, \mathrm{d}s\,, \qquad x \to 0^+, +\infty$$

Let $x = e^t$, it is equivalent to find the behavior as $t \to \pm\infty$.

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \varphi(s) e^{-ts}\, \mathrm{d}s\,, \qquad t \to -\infty, +\infty$$

When $t \to +\infty$, we have $h(s) = -s$, and γ_v point to the right. When $t \to -\infty$, γ_v to the left. We can use Mellin transform to calculate the sum of a series.

$$F(x) = \sum_{k\geq 1} \lambda_k f(\mu_k x), \qquad \lambda_k \in \mathbb{C},\ \ \mu_k > 0$$

First, we obtain the Mellin transform for $f(\mu_k x)$, which is $\varphi(s)/\mu_k^s$ for $s \in \Omega_{ab}$. The Mellin transform of $F(x)$ is

$$\mathcal{M}\{F(x)\} = \Lambda(s)\varphi(s), \qquad \Lambda(s) = \sum_{k\geq 1} \lambda_k \mu_k^{-s}, \qquad s \in \{s \in \mathbb{C} \mid \operatorname{Re} s > \sigma_0\}$$

The series $\Lambda(s)$ is called the general Dirichlet series. Finally, the inverse transform gives

$$F(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(s)\varphi(s)x^{-s}\,\mathrm{d}s, \qquad c \in (\max\{a, \sigma_0\}, b)$$

Example 6: Harmonic sum

$$h(x) = \sum_{k\geq 1}\left(\frac{1}{k} - \frac{1}{k+x}\right) = \sum_{k\geq 1}\frac{1}{k}\cdot\frac{x/k}{1+x/k}, \qquad x \to +\infty$$

We can choose the following λ_k, μ_k and $f(x)$

$$\lambda_k = \mu_k = \frac{1}{k}, \qquad f(x) = \frac{x}{1+x}, \qquad h(x) = \sum_{k\geq 1}\lambda_k f(\mu_k x)$$

The Mellin transform $\varphi(s)$ is calculated as

$$\varphi(s) = \int_0^\infty \frac{x^s}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin \pi(s+1)} = -\frac{\pi}{\sin \pi s}, \qquad -1 < \operatorname{Re} s < 0$$

The general Dirichlet series is

$$\Lambda(s) = \sum_{k\geq 1}\lambda_k \mu_k^{-s} = \sum_{k\geq 1}\frac{1}{k^{1-s}} = \zeta(1-s), \qquad \operatorname{Re} s < 0$$

Take $c \in (-1, 0)$, we have

$$h(x) = -\frac{1}{2i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(1-s)}{\sin \pi s}x^{-s}\,\mathrm{d}s, \qquad x \to +\infty$$

We recognize that $s = 0$ is a pole of order 2, while $s = n$ are simple poles for $n \geq 1$.

$$h(x) \sim \pi\left[\operatorname{Res}(\Phi(s), 0) + \sum_{n\geq 1}\operatorname{Res}(\Phi(s), n)\right], \qquad \Phi(s) = \frac{\zeta(1-s)}{\sin \pi s}x^{-s}, \qquad x \to +\infty$$

The residues of the simple poles are calculated as

$$\operatorname{Res}(\Phi(s), n) = \zeta(1-n)x^{-n}\lim_{s\to n}\frac{s-n}{\sin \pi s} = \frac{(-1)^n}{\pi}\zeta(1-n)x^{-n}, \qquad n \geq 1$$

For the residue at $s = 0$, based on the following result

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

The Taylor series can be calculated as

$$\Phi(s) = \frac{1}{\pi s + \cdots} \cdot \left(-\frac{1}{s} + \gamma + \cdots\right) \cdot (1 - s \ln x + \cdots) = \frac{1}{\pi s}(\ln x + \gamma) + \cdots$$

Therefore, the residue is

$$\operatorname{Res}(\Phi(s), 0) = \frac{\ln x + \gamma}{\pi}$$

The asymptotic expansion of $h(x)$ is

$$h(x) \sim \ln x + \gamma + \sum_{n \geq 1} (-1)^n \zeta(1-n) x^{-n}$$

$$= \ln x + \gamma + \frac{1}{2x} - \sum_{n \geq 2} \frac{(-1)^n B_n}{n} \frac{1}{x^n}, \qquad x \to +\infty$$

Take $x = n$, we have $h(n) = H_n$, which recovers the harmonic sum.

Example 7: Mellin transform for summation

$$F(x) = \sum_{k \geq 1} e^{-\sqrt{k}x}, \qquad x \to 0^+$$

We can choose the following λ_k, μ_k and $f(x)$

$$\lambda_k = 1, \qquad \mu_k = \sqrt{k}, \qquad f(x) = e^{-x}, \qquad F(x) = \sum_{k \geq 1} \lambda_k f(\mu_k x)$$

The Mellin transform $\varphi(s)$ is

$$\varphi(s) = \int_0^\infty e^{-x} x^{s-1} \, \mathrm{d}x = \Gamma(s), \qquad \operatorname{Re} s > 0$$

The general Dirichlet series is

$$\Lambda(s) = \sum_{k \geq 1} \lambda_k \mu_k^{-s} = \sum_{k \geq 1} k^{-s/2} = \zeta\left(\frac{s}{2}\right), \qquad \operatorname{Re} s > 2$$

Take $c > 2$, we have

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi(s) \, \mathrm{d}s, \qquad \Phi(s) = \Gamma(s) \zeta\left(\frac{s}{2}\right) x^{-s}, \qquad x \to 0^+$$

The steepest descent paths point to the left in this case. We obtain

$$F(x) \sim \operatorname{Res}(\Phi(s), 2) + \sum_{n \geq 0} \operatorname{Res}(\Phi(s), -n), \qquad x \to 0^+$$

All are simple poles. The residue at $s = 2$ is

$$\operatorname{Res}(\Phi(s), 2) = \frac{\Gamma(2)}{x^2} \lim_{s \to 2} (s - 2) \zeta\left(\frac{s}{2}\right) = \frac{2}{x^2}$$

The residues at $s = -n$ are

$$\operatorname{Res}(\Phi(s), -n) = \zeta\left(-\frac{n}{2}\right) x^n \lim_{s \to -n} (s + n) \Gamma(s) = \frac{(-1)^n}{n!} \zeta\left(-\frac{n}{2}\right) x^n, \qquad n \geq 0$$

The asymptotic expansion of $F(x)$ is

$$F(x) \sim \frac{2}{x^2} + \sum_{n \geq 0} \frac{(-1)^n}{n!} \zeta\left(-\frac{n}{2}\right) x^n, \qquad x \to 0^+$$

Example 8: Polylogarithm

$$\operatorname{Li}_k(z) = \sum_{n \geq 1} \frac{z^n}{n^k}, \qquad k \in \mathbb{N}^*, \qquad \operatorname{Li}_1(z) = -\ln(1 - z)$$

The radius of convergence is $R = 1$. As $z \to 1^-$, it converges when $k \geq 2$. Consider

$$F(x) = \operatorname{Li}_k(e^{-x}), \qquad x \to 0^+, \quad z = e^{-x} \to 1^-$$

We can choose the following λ_k, μ_k and $f(x)$

$$\lambda_n = \frac{1}{n^k}, \qquad \mu_n = n, \qquad f(x) = e^{-x}, \qquad F(x) = \sum_{n \geq 1} \lambda_n f(\mu_n x)$$

Then we obtain

$$\varphi(s) = \Gamma(s), \qquad \Lambda(s) = \sum_{n \geq 1} \lambda_n \mu_n^{-s} = \zeta(k + s), \qquad \operatorname{Re} s > 0$$

Take $c > 0$, we have

$$F(x) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \Phi(s) \, \mathrm{d}s, \qquad \Phi(s) = \Gamma(s) \zeta(k + s) x^{-s}, \qquad x \to 0^+$$

There is a pole of order 2 at $s = 1 - k$ and other simple poles at $s = -n$ when $n \neq k - 1$. The residues can be obtained as

$$\operatorname{Res}(\Phi(s), -n) = \frac{(-1)^n}{n!} \zeta(k - n) x^n, \qquad n \neq k - 1$$

$$\operatorname{Res}(\Phi(s), 1 - k) = \frac{(-1)^{k-1}}{(k - 1)!} x^{k-1} (H_{k-1} - \ln x)$$

The asymptotic expansion of $\operatorname{Li}_k(e^{-x})$ is

$$\operatorname{Li}_k(e^{-x}) \sim \frac{(-1)^{k-1}}{(k - 1)!} (H_{k-1} - \ln x) x^{k-1} + \sum_{\substack{n=0 \\ n \neq k-1}}^{\infty} \frac{(-1)^n}{n!} \zeta(k - n) x^n, \qquad x \to 0^+$$

Example 9: Partition function in number theory

The partition function p_n denotes the number of possible partitions of a non-negative integer n. As $n \to +\infty$, the asymptotic expansion of p_n is

$$p_n = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}\big(1 + o(1)\big)$$

The generating function $f(z)$ is

$$f(z) = 1 + \sum_{n\geq 1} p_n z^n = \prod_{m\geq 1} \frac{1}{1 - z^m}, \qquad |z| < 1$$

From the Z transform, we can write

$$p_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z^{n+1}}\,\mathrm{d}z, \qquad \gamma: z = \rho e^{i\theta}, \qquad 0 < \rho < 1$$

However, the poles of $f(z)$ on $|z| = 1$ are dense, which makes it hard for $\rho \to 1$. Consider

$$L(x) = \ln f(e^{-x}) = -\sum_{m\geq 1} \ln(1 - e^{-mx}), \qquad x \to 0^+, \quad z = e^{-x} \to 1^-$$

We can choose the following λ_k, μ_k and $g(x)$

$$\lambda_m = -1, \qquad \mu_m = m, \qquad g(x) = \ln(1 - e^{-x}), \qquad L(x) = \sum_{m\geq 1} \lambda_m g(\mu_m x)$$

Then we obtain

$$\varphi(s) = \int_0^\infty \ln(1 - e^{-x})\, x^{s-1}\,\mathrm{d}x, \qquad \Lambda(s) = \sum_{m\geq 1} \lambda_m \mu_m^{-s} = -\zeta(s)$$

The Mellin transform is obtained as

$$\ln(1 - e^{-x}) = -\sum_{l\geq 1} \frac{e^{-lx}}{l}, \qquad \varphi(s) = -\sum_{l\geq 1} \frac{1}{l}\int_0^\infty e^{-lx} x^{s-1}\,\mathrm{d}x = -\zeta(s+1)\Gamma(s)$$

Take $c > 1$, we have

$$L(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)\,\mathrm{d}s, \qquad \Phi(s) = \zeta(s)\zeta(s+1)\Gamma(s)x^{-s}, \qquad x \to 0^+$$

After analyzing the poles, and the zeros of $\zeta(s)$ and $\zeta(s+1)$, we have $s = 0$ being a pole of order 2, while $s = \pm 1$ being simple poles. The contour can be chosen to close at $s = -c$, i.e., a rectangular contour. Using the following equations

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s)\Gamma(1-s)\sin\frac{\pi s}{2}$$

$$\zeta(s+1) = 2^{s+1}\pi^s \zeta(-s)\Gamma(-s)\cos\frac{\pi s}{2}$$

$$\Gamma(s) = \pi[\sin(\pi s)\,\Gamma(1-s)]^{-1}$$

We can obtain an equation for $\widetilde{\Phi}(s)$, which is

$$\widetilde{\Phi}(s) = (2\pi)^{2s}\widetilde{\Phi}(-s), \qquad \widetilde{\Phi}(s) = \zeta(s)\zeta(s+1)\Gamma(s)$$

After calculating the residues, and study the relation between the integrals along $\operatorname{Re} s = \pm c$, we reach an equation for $L(x)$

$$L(x) = \frac{\pi^2}{6x} + \frac{1}{2}\ln\left(\frac{x}{2\pi}\right) - \frac{x}{24} + L\left(\frac{4\pi^2}{x}\right)$$

This implies that $L(x)$ is close to a modular form.

- Exercise

Steepest descents

$$F(\lambda) = \int_i^1 e^{-\lambda z^2}\,\mathrm{d}z\,, \qquad \lambda \to \infty$$

The steepest descent path is $\tilde{\gamma}: (\gamma_v: i \to 0) \cup (\gamma_v: 0 \to 1)$. On the two segments, we have

$$I_1 = -i\int_0^1 e^{\lambda t^2}\mathrm{d}t\,, \qquad I_2 = \int_0^1 e^{-\lambda t^2}\mathrm{d}t$$

The dominant contribution arises from the endpoint $t = 1$ in I_1. Denote

$$R_{\max} = R(1) = 1, \qquad \tilde{R}(t) = 1 - t^2 = s, \qquad t = \sqrt{1-s}$$

We first obtain the following Taylor series

$$\phi(s) = \frac{1}{\sqrt{1-s}} = 1 + \frac{1}{2}s + \frac{3}{8}s^2 + \frac{5}{16}s^3 + O(s^4)$$

The integral becomes

$$I_1 \sim \frac{e^\lambda}{2i}\int_0^1 \frac{e^{-\lambda s}}{\sqrt{1-s}}\,\mathrm{d}s = \frac{e^\lambda}{2i}\int_0^1 e^{-\lambda s}\phi(s)\,\mathrm{d}s = \frac{e^\lambda}{2i}\sum_{n=0}^{\infty}\frac{\phi^{(n)}(0)}{\lambda^{n+1}}, \qquad \lambda \to \infty$$

Based on the Taylor coefficients, we have

$$F(\lambda) \sim \frac{e^\lambda}{2i\lambda}\left(1 + \frac{1}{2}\lambda^{-1} + \frac{3}{4}\lambda^{-2} + \frac{15}{8}\lambda^{-3} + O(\lambda^{-4})\right), \qquad \lambda \to +\infty$$

Hankel function

$$H_\nu^{(1)}(\lambda) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{\lambda\sinh(z) - \nu z}\,\mathrm{d}z\,, \qquad \operatorname{Re}\lambda > 0$$

In specific, we want to study the asymptotic behavior as $\lambda \to +\infty$ of the scaled Hankel function

$$H_{c\lambda}^{(1)}(\lambda) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{\lambda\sinh(z) - c\lambda z}\,\mathrm{d}z\,, \qquad \lambda \to +\infty, \qquad c = \cosh\alpha > 1$$

We recognize $h(z) = \sinh z - z\cosh\alpha$. The saddle points are calculated as

$$h'(z) = \cosh z - \cosh\alpha = 0, \qquad z = \pm\alpha + 2\pi k i, \qquad \alpha > 0, \qquad k \in \mathbb{Z}$$

The real and imaginary parts are

$$u(x,y) = \sinh x \cos y - x\cosh\alpha\,, \qquad v(x,y) = \cosh x \sin y - y\cosh\alpha$$

We notice the following contour levels

$$\gamma_u\colon \{\operatorname{Re} z = 0,\ \ u = 0\}, \qquad \gamma_v\colon \{\operatorname{Im} z = k\pi,\ \ k \in \mathbb{Z},\ \ v = -k\pi\cosh\alpha\}$$

Therefore, we can construct the steepest descent path as

$$\tilde{\gamma}\colon (\gamma_v\colon -\infty \to 0) \cup (\gamma_u\colon 0 \to \pi i) \cup (\gamma_v\colon \pi i \to \infty + \pi i)$$

There is one saddle point $z_0 = -\alpha$ on the first γ_v segment where u is maximal along the path. Note that $g(z) = 1$, and we also have

$$h(z_0) = -\sinh\alpha + \alpha\cosh\alpha\,, \qquad h''(z_0) = -\sinh\alpha\,, \qquad \arg z'(t_0) = 0$$

From the following equation

$$F(\lambda) \sim e^{\lambda h(z_0)}\left[e^{i\arg z'(t_0)}\sqrt{\frac{2\pi}{|h''(z_0)|}}\,g(z_0)\frac{1}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right)\right], \qquad \lambda \to +\infty$$

The leading term of the scaled Hankel function is

$$H_{c\lambda}^{(1)}(\lambda) = -ie^{\lambda(\alpha\cosh\alpha - \sinh\alpha)}\sqrt{\frac{2}{\pi\lambda\sinh\alpha}}\big(1 + O(\lambda^{-1})\big), \qquad \lambda \to +\infty$$

If we denote $\nu = c\lambda = \lambda\cosh\alpha$, the expression becomes

$$H_\nu^{(1)}(\nu\operatorname{sech}\alpha) = -ie^{\nu(\alpha - \tanh\alpha)}\sqrt{\frac{2}{\pi\nu\tanh\alpha}}\big(1 + O(\nu^{-1})\big), \qquad \nu \to +\infty$$

Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C t^{-z}e^t\,\mathrm{d}t\,, \qquad C\colon (-\infty - i\varepsilon \to 0 \to -\infty + i\varepsilon)$$

This provides an analytic continuation of $\Gamma^{-1}(z)$. The branch cut is along the negative real axis with $\arg t \in (-\pi, \pi)$. We want to study the asymptotic behavior for $z = \lambda e^{i\kappa}$ as $\lambda \to +\infty$ when $\kappa \in (-\pi, \pi)$, which is the Stirling formula. With the transform $t = \lambda\tau$, we have

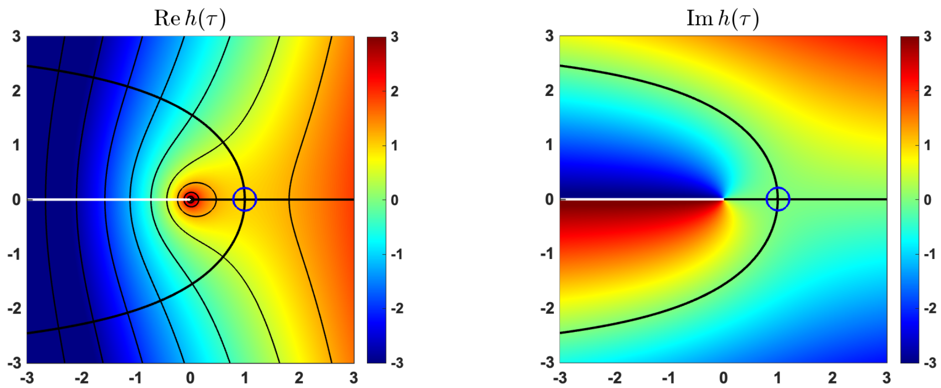
$$\frac{1}{\Gamma(z)} = \frac{\left(ze^{-i\kappa}\right)^{-z+1}}{2\pi i}\int_C e^{\lambda(\tau - e^{i\kappa}\ln\tau)}\,\mathrm{d}\tau\,, \qquad h(\tau) = \tau - e^{i\kappa}\ln\tau$$

The saddle point is obtained as

$$h'(\tau) = 1 - \frac{e^{i\kappa}}{\tau} = 0, \qquad \tau_0 = e^{i\kappa}, \qquad h(\tau_0) = e^{i\kappa}(1 - i\kappa)$$

The second-order derivative and the descent angle (see note below) are

$$h''(\tau_0) = e^{-i\kappa}, \qquad \arg z'(t_0) = \frac{\kappa + \pi}{2}$$



An example of the contour levels for $\kappa = 0$ and the saddle point on γ_v are shown above. From the upper and lower side of the branch cut, we can use γ_u: $\{u < 0\}$ to connect to the steepest descent path γ_v: $\{v = v_0\}$ that goes through the saddle point. Therefore, the integral becomes

$$\int_C e^{\lambda(\tau - e^{i\kappa}\ln\tau)}\,\mathrm{d}\tau \sim e^{z(1-i\kappa)}\left[ie^{\frac{i\kappa}{2}}\sqrt{\frac{2\pi}{ze^{-i\kappa}}} + o\left(\frac{1}{\sqrt{\lambda}}\right)\right]$$

$$= ie^{z}e^{i\kappa(1-z)}\sqrt{\frac{2\pi}{z}}\left(1 + O(\lambda^{-1})\right), \qquad \lambda \to +\infty$$

The leading term of the asymptotic expansion is

$$\frac{1}{\Gamma(z)} \sim \frac{z^{-z+1/2}e^{z}}{\sqrt{2\pi}}\left(1 + O(\lambda^{-1})\right)$$

Hence, we derive the following limit

$$\lim_{\substack{|z|\to+\infty \\ \kappa\in(-\pi,\pi)}} \frac{\Gamma(z)}{\sqrt{2\pi}\, z^{z-1/2}e^{-z}} = 1$$

Note. The descent angle is obtained from the Taylor series of $h(z)$ around the saddle point z_0

$$h(z) - h(z_0) = \frac{h^{(p)}(z_0)}{p!}(z - z_0)^p + O\left((z - z_0)^{p+1}\right), \qquad h^{(p)}(z_0) = \left|h^{(p)}(z_0)\right|e^{i\alpha}$$

The imaginary part gives $v(z) = v(x, y)$, which is

$$v(z) - v(z_0) = \frac{\left|h^{(p)}(z_0)\right|}{p!}r^p\sin(p\theta + \alpha) + O(r^{p+1}), \qquad z - z_0 = re^{i\theta}$$

There are $2p$ curves of γ_v: $\{v = v_0\}$ emanating from z_0 with the tangent directions

$$\theta_v = \frac{n\pi - \alpha}{p}, \qquad n = 0,1,\cdots,2p-1$$

The descent angle is one of them and can be analyzed from the contour levels.

Effect of branch point

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda x^2} e^{4i\lambda x} \ln[(x - x_0)^2 + 1] \, \mathrm{d}x$$

We recognize the functions $h(z)$ and $g(z)$ as

$$h(z) = -z^2 + 4iz = -(z - 2i)^2 - 4, \qquad g(z) = \ln[(z - x_0)^2 + 1]$$

The real and imaginary parts of $h(z)$ are

$$u(x, y) = -x^2 + y^2 - 4y, \qquad v(x, y) = -2x(y - 2)$$

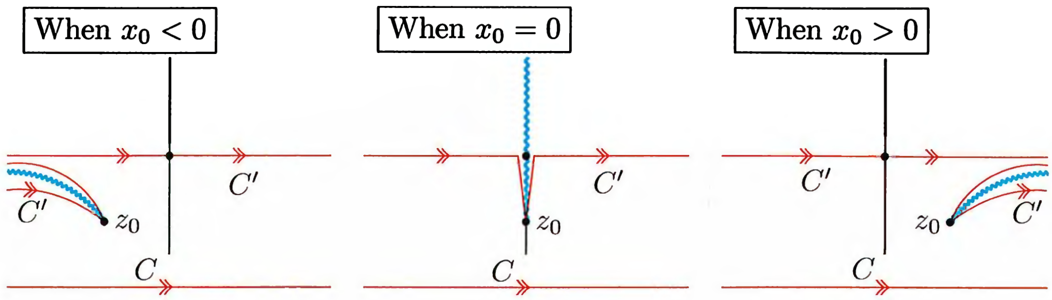
Note that γ_v: $\{\mathrm{Im}\, z = 2,\ v = 0\}$ is a steepest descent path, and there lies a saddle point of $h(z)$

$$z_s = 2i, \qquad h(z_s) = -4, \qquad h''(z_s) = -2$$

The branch point of $g(z)$ in the upper half plane that lies between $\mathbb{R}$ and γ_v is

$$z_0 = x_0 + i, \qquad h(z_0) = -x_0^2 - 3 + 2ix_0, \qquad h'(z_0) = -2(x_0 - i)$$

The deformed path is the same as in a previous example. For $x_0 \neq 0$, the branch cut in the steepest descent direction is a hyperbola, while for $x_0 = 0$ it is the imaginary axis.



Possible asymptotic contributions to the integral come from either the saddle point z_s or the branch point z_0. Comparing the real part $u(x, y)$ at these two points, we have

$$u(z_s) = -4, \qquad u(z_0) = -x_0^2 - 3$$

Hence, when $|x_0| < 1$ the branch point dominate, while when $|x_0| > 1$ the saddle point does. When $|x_0| = 1$, both contributions may be important. For the special case $x_0 = 0$, the branch point still dominates.

To obtain the branch point contribution, note that $\arg[(z - z_0)^2 + 1]$ is larger on the right side, so we always have the following sum of contributions from the two arcs along the branch cut

$$F_0(\lambda) = 2i\pi \int_{z_0}^{\infty} e^{\lambda h(z)} \, \mathrm{d}z = 2i\pi e^{\lambda h(z_0)} \int_0^{+\infty} e^{-\lambda t} z'(t) \, \mathrm{d}t$$

There will be a transform $z = z(t)$ such that $h(z) - h(z_0) = -t$ on the branch cut. The leading order term only needs $z'(0) = -1/h'(z_0)$. Hence, we obtain

$$F_0(\lambda) \sim -\frac{\pi e^{2i\lambda x_0}}{\lambda(1 + ix_0)} e^{-\lambda(x_0^2+3)}\left(1 + O(\lambda^{-1})\right), \qquad \lambda \to +\infty$$

The saddle point contribution is calculated as

$$F_s(\lambda) \sim e^{\lambda h(z_s)}\left[e^{i\arg z'(t_s)}\sqrt{\frac{2\pi}{|h''(z_s)|}}\,g(z_s)\frac{1}{\sqrt{\lambda}}+o\left(\frac{1}{\sqrt{\lambda}}\right)\right]$$
$$\sim e^{-4\lambda}\sqrt{\frac{\pi}{\lambda}}\ln(x_0^2-3-4ix_0)\left(1+O(\lambda^{-1})\right),\qquad \lambda\to+\infty$$

Comparing these two contributions, the saddle point is more important at $|x_0|=1$, because it involves $\lambda^{-1/2}$ instead of λ^{-1} in addition to the same exponential decay.

Inverse Fourier transform

$$f(x)=\int_{-\infty}^{+\infty}\frac{k^2-3k+4}{[(k-2)^2+4]^2[(k+3)^2+1]}e^{ikx}\,\mathrm{d}k\,,\qquad x<0$$

The integrand has a simple pole at $k_1=-3-i$ and a pole of order 2 at $k_2=2-2i$. We can obtain the residues at these two poles as

$$\mathrm{Res}(g,k_1)=e^xe^{-3ix}\frac{3(-3+7i)}{8(14+5i)^2}=A_1e^xe^{-3ix},\qquad \mathrm{Res}(g,k_2)=A_2e^{2x}e^{2ix}$$

Therefore, the exact result of the integral is

$$f(x)=2\pi i\left(A_1e^xe^{-3ix}+A_2e^{2x}e^{2ix}\right),\qquad x<0$$

As $x\to-\infty$, the asymptotic behavior is

$$f(x)\sim 2\pi iA_1e^xe^{-3ix}+O(e^{2x}),\qquad x\to-\infty$$

Inverse Laplace transform

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\tanh s}{s}e^{st}\,\mathrm{d}s$$

The simple poles of the function are

$$s^*=0,\qquad s_k=\frac{2k+1}{2}\pi i,\qquad k\in\mathbb{Z}$$

The residues at these poles are obtained as

$$\mathrm{Res}(g,0)=0,\qquad \mathrm{Res}(g,s_k)=\frac{1}{s_k}e^{s_kt}$$

The inverse Laplace transform is obtained as the following exact result

$$f(t)=-\sum_{k\in\mathbb{Z}}\frac{2i}{(2k+1)\pi}e^{\frac{(2k+1)\pi i}{2}t}$$

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{-\sqrt{s}}}{\sqrt{s}} e^{st}\,\mathrm{d}s$$

The branch cut is on the negative real axis. The integral then becomes

$$f(t) = \frac{1}{2\pi i}\left(\int_{-\infty}^{0} \frac{e^{i\sqrt{-s}}}{-i\sqrt{-s}} e^{st}\,\mathrm{d}s + \int_{0}^{-\infty} \frac{e^{-i\sqrt{-s}}}{i\sqrt{-s}} e^{st}\,\mathrm{d}s\right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{+\infty} \frac{e^{i\sqrt{x}}}{\sqrt{x}} e^{-xt}\,\mathrm{d}x + \int_{0}^{+\infty} \frac{e^{-i\sqrt{x}}}{\sqrt{x}} e^{-xt}\,\mathrm{d}x\right) = \frac{1}{\pi}\int_{0}^{+\infty} \frac{\cos\sqrt{x}}{\sqrt{x}} e^{-xt}\,\mathrm{d}x$$

$$= \frac{2}{\pi}\int_{0}^{+\infty} e^{-tu^2}\cos u\,\mathrm{d}u = \frac{2}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_{0}^{+\infty} e^{-tu^2}u^{2n}\,\mathrm{d}u$$

$$= \frac{1}{\sqrt{\pi t}}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^{2n}n!}\frac{1}{t^n} = \frac{1}{\sqrt{\pi t}}e^{-1/4t}$$

The inverse Laplace transform is also exact.

Asymptotic Analysis of Integrals (4): Stationary Phase

We analyze the asymptotic expansion of the following integral

$$F(\lambda) = \int_a^b e^{i\lambda I(t)} g(t)\,\mathrm{d}t\,, \qquad \lambda \to +\infty$$

With the interval $[a, b] \subseteq \mathbb{R}$, functions $I: [a, b] \to \mathbb{R}$ and $g: [a, b] \to \mathbb{C}$. When $\lambda \to -\infty$, we can take the complex conjugate of the integral. For the method of steepest descent, with require the functions $I(t)$ and $g(t)$ to be analytic satisfying Cauchy-Riemann equation. However, we now only require them to be C^∞. The asymptotic analysis can only be performed on $[a, b]$, and we cannot deform the integration path.

- Oscillatory integral (5.1)

Example: Fresnel integral

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \int_0^\infty \cos(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}}, \qquad I = \int_0^\infty e^{ix^2}\mathrm{d}x = e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

For an integral with parameter, we have

$$\int_0^\infty e^{i\lambda x^2}\mathrm{d}x = \frac{1}{2}e^{i\pi/4}\sqrt{\frac{\pi}{\lambda}} = O\left(\frac{1}{\sqrt{\lambda}}\right)$$

- Nonlocal contributions (5.2)

Nonlocal contributions arise from $I'(t) \neq 0$. Assume $[t_1, t_2] \subseteq [a, b]$ such that $I \in C^1[t_1, t_2]$ and $I'(t) \neq 0$, i.e. $I(t)$ monotonically increases or decreases. For the interval $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} e^{i\lambda I(t)} g(t)\,\mathrm{d}t = \frac{1}{i\lambda}\left[e^{i\lambda I(t)}\frac{g(t)}{I'(t)}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} e^{i\lambda I(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{g(t)}{I'(t)}\,\mathrm{d}t\right]$$

With integration by part, we obtain a factor $(i\lambda)^{-1}$. The bracket term is bounded by a constant independent of λ, which gives

$$\left|\int_{t_1}^{t_2} e^{i\lambda I(t)} g(t)\,\mathrm{d}t\right| \leq \frac{1}{i\lambda}\left(\left|\frac{g(t_1)}{I'(t_1)}\right| + \left|\frac{g(t_2)}{I'(t_2)}\right| + \int_{t_1}^{t_2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\frac{g(t)}{I'(t)}\right|\mathrm{d}t\right) = O\left(\frac{1}{\lambda}\right)$$

More generally, with $g_0(t) = g(t)$ we can derive an asymptotic expansion as

$$\int_{t_1}^{t_2} e^{i\lambda I(t)} g(t)\,\mathrm{d}t \sim \sum_{n\geq 0}\left[\frac{e^{i\lambda I(t)}}{(i\lambda)^{n+1}}\frac{g_n(t)}{I'(t)}\right]_{t_1}^{t_2}, \qquad g_n(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{g_{n-1}(t)}{I'(t)}$$

Compared with Laplace's method in which the nonlocal contributions is $O(e^{-c\lambda})$. Now for the method of stationary phase, the nonlocal contributions are larger.

- Contributions from interior stationary phase points (5.3)

Consider $I \in C^1[a,b]$ with a stationary phase point $t_0 \in (a,b)$ and $I'(t_0) = 0$. We require the point t_0 as non-degenerate, stated as there exists $\delta > 0$ such that $I \in C^{2N+1}[t_0 - 2\delta, t_0 + 2\delta]$ and $I''(t_0) \neq 0$. We also require $g \in C^{2N}$ and the support of g is within the interval.

$$F_\delta(\lambda) = \int_{t_0-2\delta}^{t_0+2\delta} e^{i\lambda I(t)} g(t)\,\mathrm{d}t, \qquad \lambda \to +\infty$$

Around $t = t_0$, the Taylor series of $I(t)$ is

$$I(t) = I(t_0) + \frac{1}{2} I''(t_0)(t - t_0)^2 + \cdots$$

Denote $\sigma = \operatorname{sgn} I''(t_0)$. According to Morse Lemma, we find a transformation such that

$$I(t) = I(t_0) + \sigma s^2, \qquad F_\delta(\lambda) = e^{i\lambda I(t_0)} \int_{s_-}^{s_+} e^{i\lambda\sigma s^2} g\big(t(s)\big) t'(s)\,\mathrm{d}s$$

The lower and upper limits of the integral are calculated as

$$s_\pm = \pm\sqrt{\sigma[I(t_0 \pm 2\delta) - I(t_0)]}, \qquad t(s), s(t) \in C^{2N+1}, \qquad t'(s) \in C^{2N}$$

Then we have $k(s) = g(t(s))\, t'(s) \in C^{2N}$ and we want to approximate $k(s)$ by a polynomial. We first have the Taylor series

$$k(s) = Q(s) + O(s^{2N}), \qquad Q(s) = \sum_{m=0}^{2N-1} \frac{k^{(m)}(0)}{m!} s^m$$

However, $k(s)$ does not vanish near $s_\pm$. We modify it by constructing $R(s)$ as

$$R_1(s) = [(s_+ - s)(s - s_-)]^N, \qquad \frac{1}{R_1(s)} = R_2(s) + O(s^{2N}), \qquad R(s) = R_1(s) R_2(s)$$

The polynomial cutoff function $R(s)$ has the following properties

$$R(s) = 1 + O(s^{2N}), \qquad R(s_\pm) = \cdots = R^{(N-1)}(s_\pm) = 0$$

The polynomial approximation of $k(s)$ is constructed as

$$P(s) = Q(s)R(s), \qquad P(s) = Q(s) + O(s^{2N}) = \sum_{m=0}^{2N-1} \frac{k^{(m)}(0)}{m!} s^m + O(s^{2N})$$

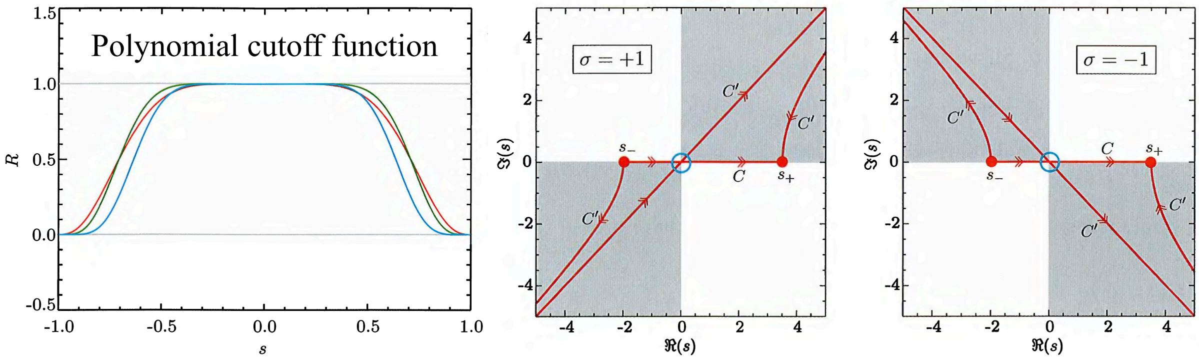
This also implies that $s_\pm$ are the zeros of $P(s)$ of order N and $\deg P = 6N - 2$. Denote

$$J_1 = \int_{s_-}^{s_+} e^{i\lambda\sigma s^2} P(s)\,\mathrm{d}s, \qquad J_2 = \int_{s_-}^{s_+} e^{i\lambda\sigma s^2} [k(s) - P(s)]\,\mathrm{d}s$$

We apply the method of steepest descent for J_1 as all functions are analytic. For $\sigma = 1$, we have

$$h(s) = is^2, \qquad u = -2xy, \qquad v = x^2 - y^2$$

The steepest descent path contributes to the integral at the saddle point $s = 0$, and also at the endpoints $s = s_\pm$. However, we have $P(s_\pm) = \cdots = P^{(N-1)}(s_\pm) = 0$, so for the first N terms



in the asymptotic expansion, the only contribution is from the saddle point. Along the diagonal that goes through the saddle point $s = 0$, we have $s = ue^{i\sigma\pi/4}$ and the integral becomes

$$J_1 = \int_{-\infty}^{+\infty} e^{-\lambda u^2} P\left(ue^{\frac{i\sigma\pi}{4}}\right) e^{\frac{i\sigma\pi}{4}}\,\mathrm{d}u = \sum_{n=0}^{2N-1} \frac{k^{(n)}(0)}{n!} e^{\frac{i(n+1)\sigma\pi}{4}} \int_{-\infty}^{+\infty} e^{-\lambda u^2} u^n\,\mathrm{d}u$$

$$= \sum_{n=0}^{N-1} \frac{\sqrt{\pi} e^{i(2n+1)\sigma\pi/4}}{2^{2n} n!} \frac{k^{(2n)}(0)}{\lambda^{n+1/2}} + O\left(\frac{1}{\lambda^{N+1/2}}\right), \qquad \lambda \to +\infty$$

Due to symmetry, only even order terms are retained. The term J_2 can be analyzed from integration by parts. We define a sequence of functions

$$w_0(s) = k(s) - P(s) = O(s^{2N}), \qquad w_n(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{w_{n-1}}{s}\right)$$

At $s = s_\pm$, we have $w_0 = \cdots = w_{N-1} = 0$, so the boundary terms are zero. We can obtain

$$J_2 = \frac{1}{(2i\sigma\lambda)^N} \int_{s_-}^{s_+} e^{i\lambda\sigma s^2} w_N(s)\,\mathrm{d}s\,, \qquad |J_2| \le \frac{1}{(2\lambda)^N} \int_{s_-}^{s_+} |w_N(s)|\,\mathrm{d}s = O\left(\frac{1}{\lambda^N}\right)$$

The final result for $F_\delta(\lambda)$ becomes

$$F_\delta(\lambda) = \int_{t_0^-}^{t_0^+} e^{i\lambda I(t)} g(t)\,\mathrm{d}t = e^{i\lambda I(t_0)} e^{\frac{i\sigma\pi}{4}} \sum_{n=0}^{N-1} \frac{i^{n\sigma}\sqrt{\pi}}{2^{2n} n!} \frac{k^{(2n)}(0)}{\lambda^{n+1/2}} + O\left(\frac{1}{\lambda^N}\right), \qquad \lambda \to +\infty$$

If $I, g \in C^\infty$, then we can take N arbitrarily large. To obtain the leading order term, note that

$$k(0) = g(t_0) t'(0) = g(t_0)\sqrt{\frac{2\sigma}{I''(t_0)}} = g(t_0)\sqrt{\frac{2}{|I''(t_0)|}}$$

We thus derive the practical result

$$F_\delta(\lambda) = e^{i\lambda I(t_0)} e^{\frac{i\sigma\pi}{4}} \sqrt{\frac{2\pi}{\lambda |I''(t_0)|}}\, g(t_0) + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty$$

➢ Generic leading-order behavior (5.4)

If there are no stationary phase points within $[a, b]$, then we only need the contributions from the endpoints. The result is

$$F(\lambda) = \frac{1}{i\lambda}\left[e^{i\lambda I(b)}\frac{g(b)}{I'(b)} - e^{i\lambda I(a)}\frac{g(a)}{I'(a)}\right] + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty$$

If there are M stationary phase points $t_1, \cdots, t_M$, then we have

$$F(\lambda) = \sum_{k=1}^{M} e^{i\lambda I(t_k)} e^{\frac{i\sigma_k \pi}{4}} \sqrt{\frac{2\pi}{\lambda |I''(t_k)|}}\, g(t_k) + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty$$

Every t_k contributes to the same $\lambda^{-1/2}$ order, which is different from the Laplace's method.

Example 1: Bessel function

For integer order $n \in \mathbb{Z}$, the Bessel function can be defined as

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - x\sin t)\ \mathrm{d}t = \mathrm{Re}\left[\frac{1}{\pi}\int_0^{\pi} e^{i(nt - x\sin t)}\mathrm{d}t\right], \qquad x \to +\infty$$

We recognize

$$I(t) = -\sin t, \qquad g(t) = e^{int}, \qquad t_0 = \frac{\pi}{2}, \qquad I(t_0) = -1, \qquad I''(t_0) = 1$$

The leading order term is obtained as

$$J_n(x) \sim \frac{1}{\pi}\mathrm{Re}\left[e^{-i\lambda} e^{\frac{i\pi}{4}} \sqrt{\frac{2\pi}{x}}\, e^{\frac{in\pi}{2}}\right] \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right), \qquad x \to +\infty$$

Denote x_{nk} as the k-th positive root of $J_n(x)$. We have

$$x_{nk} \sim k\pi + \frac{\pi}{2}\left(n - \frac{1}{2}\right), \qquad k \to +\infty$$

Example 2: Linear dispersive waves (5.5)

$$u(x,t) = \int_{\mathbb{R}} A(k) e^{i(kx - \omega t)}\, \mathrm{d}k, \qquad \omega = \omega(k)$$

Consider $x, t \to \infty$ with $x = v_g t$, we analyze the integral

$$F(t) = \int_{\mathbb{R}} A(k) e^{i\left(kv_g - \omega(k)\right)t}\, \mathrm{d}k, \qquad t \to +\infty$$

We recognize

$$I(k) = kv_g - \omega(k), \qquad g(k) = A(k), \qquad I'(k) = v_g - \omega'(k) = 0$$

This gives rise to the group velocity $v_g = \omega'(k)$.

Example 3: Schrödinger equation for a free particle

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}, \qquad \omega = \frac{\hbar}{2m}k^2, \qquad v_p = \frac{\hbar k}{2m}, \qquad v_g = \frac{\hbar k}{m}$$

- Multidimensional oscillatory integrals (5.7)

$$F(\lambda) = \int_\Omega e^{i\lambda I(x)} g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}, \qquad \lambda \to +\infty$$

We assume $\Omega \subseteq \mathbb{R}^d$ and $I, g \in C^\infty(\Omega)$. If we have $I'(\boldsymbol{x}) = \nabla I(\boldsymbol{x}) \neq \boldsymbol{0}$ for all $\boldsymbol{x} \in \Omega$, there are no stationary phase points, and we can write the integral as

$$F(\lambda) = \frac{1}{i\lambda} \int_\Omega \frac{g(\boldsymbol{x})}{|\nabla I(\boldsymbol{x})|^2} \nabla I(\boldsymbol{x}) \cdot \nabla\left(e^{i\lambda I(x)}\right) \mathrm{d}\boldsymbol{x}$$

To apply the divergence theorem, using the identity $\nabla \cdot (f\boldsymbol{u}) = \boldsymbol{u} \cdot \nabla f + f \nabla \cdot \boldsymbol{u}$, and we have

$$F(\lambda) = \frac{1}{i\lambda} \int_\Omega \nabla \cdot \left[\frac{g(\boldsymbol{x}) e^{i\lambda I(x)}}{|\nabla I(\boldsymbol{x})|^2} \nabla I(\boldsymbol{x})\right] \mathrm{d}\boldsymbol{x} - \frac{1}{i\lambda} \int_\Omega e^{i\lambda I(x)} \nabla \cdot \left[\frac{g(\boldsymbol{x}) \nabla I(\boldsymbol{x})}{|\nabla I(\boldsymbol{x})|^2}\right] \mathrm{d}\boldsymbol{x}$$

$$= \frac{1}{i\lambda} \int_{\partial\Omega} \frac{g(\boldsymbol{x}) e^{i\lambda I(x)}}{|\nabla I(\boldsymbol{x})|^2} \nabla I(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, \mathrm{d}S - \frac{1}{i\lambda} \int_\Omega e^{i\lambda I(x)} \nabla \cdot \left[\frac{g(\boldsymbol{x}) \nabla I(\boldsymbol{x})}{|\nabla I(\boldsymbol{x})|^2}\right] \mathrm{d}\boldsymbol{x}$$

Again, we have $F(\lambda) = O(\lambda^{-1})$ and a similar expansion as the one-dimensional case

$$F(\lambda) \sim \sum_{n \geq 0} \frac{1}{(i\lambda)^{n+1}} \int_{\partial\Omega} \frac{e^{i\lambda I(x)} g_n(\boldsymbol{x})}{|\nabla I(\boldsymbol{x})|^2} \nabla I(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, \mathrm{d}S, \qquad g_n(\boldsymbol{x}) = -\nabla \cdot \frac{g_{n-1}(\boldsymbol{x}) \nabla I(\boldsymbol{x})}{|\nabla I(\boldsymbol{x})|^2}$$

Now the problem becomes analyzing the boundary integral

$$\int_{\partial\Omega} e^{i\lambda I(x)} h(\boldsymbol{x})\, \mathrm{d}S$$

The stationary phase points satisfy $\nabla I(\boldsymbol{x_0}) \parallel \boldsymbol{n}(\boldsymbol{x}_0)$, the gradient is normal to the surface.

If there is only one stationary phase point $\boldsymbol{x}_0 \in \Omega$, we have

$$F(\lambda) = \int_{|x - x_0| < 2\delta} e^{i\lambda I(x)} g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + O\left(\frac{1}{\lambda}\right)$$

For a non-degenerate Hessian matrix with $\det|I''(\boldsymbol{x}_0)| \neq 0$, from Morse Lemma there exists a local transformation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{y})$ such that

$$I(\boldsymbol{x}) = I(\boldsymbol{x}_0) + y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_d^2$$

The boundary integral then becomes

$$F(\lambda) = \int_{|y| < \delta_1} e^{i\lambda(y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_d^2)} g(\boldsymbol{x}(\boldsymbol{y})) J_x(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} + O\left(\frac{1}{\lambda}\right)$$

The leading order term is related to the following integral (with $q = d - p$)

$$\int_{\mathbb{R}^d} e^{i\lambda(y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_d^2)}\, \mathrm{d}\boldsymbol{y} = e^{\frac{i\pi}{4}(p-q)} \left(\frac{\pi}{\lambda}\right)^{d/2}$$

For multidimensional oscillatory integrals, the nonlocal contributions can be larger than those from the stationary phase points, strictly for $d \geq 2$, as shown by the asymptotic series

$$F(\lambda) \sim e^{i\lambda I(\boldsymbol{x}_0)} e^{\frac{i\pi}{4}(p-q)} \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}} \frac{g(\boldsymbol{x}_0)}{\sqrt{\det|I''(\boldsymbol{x}_0)|}} + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty$$

For higher dimensions, the cancellation from oscillations becomes less efficient, and intuitively the nonlocal contributions become important.

Example: Gauss circle problem

Consider a convex domain $\Omega \subseteq \mathbb{R}^2$ and $\partial\Omega$ is smooth (e.g., circle). For $\lambda > 0$, denote $\lambda\Omega$ as the scaled domain $\{(\lambda x, \lambda y) \mid (x, y) \in \Omega\}$. We want to know the number of integer lattice points in this domain, denoted as $\#(\lambda\Omega \cap \mathbb{Z}^2)$. With the area $S(\Omega)$, we have

$$\#(\lambda\Omega \cap \mathbb{Z}^2) = \lambda^2 S(\Omega) + E(\lambda)$$

The behavior of the remainder term $E(\lambda)$ is still an open question, which is the Gauss circle problem for Ω being a unit circle. Currently, the best result is

$$E(\lambda) = O(\lambda^\theta), \qquad \theta = \frac{517}{824} + \varepsilon \approx 0.6274\cdots$$

It is also proved that $\theta > 0.5$, which implies the conjecture

$$\limsup_{r\to+\infty} \frac{|E(r)|}{\sqrt{r}} = \infty, \qquad \theta = \frac{1}{2} + \varepsilon$$

- Exercise

Influence of integration intervals

$$F(\lambda) = \int_a^b e^{i\lambda t^2} e^{\sin t}\, \mathrm{d}t\,, \qquad \lambda \to +\infty$$

Follow the steepest descent method, we recognize

$$h(z) = iz^2, \qquad u = -2xy, \qquad v = x^2 - y^2$$

For a real value $x_0 \neq 0$, we will need the hyperbolic γ_v emanating from this point, which is

$$\gamma_v\colon\ x^2 - y^2 = x_0^2$$

When $x_0 > 0$ we take the segment with $x \in [x_0, +\infty)$, while when $x_0 < 0$ we take the segment with $x \in (-\infty, x_0]$. At $x_0 = 0$, the contour line to use is $\gamma_v\colon \{y = x\}$. Now we can evaluate the integral along the hyperbola. For $x_0 = -a < 0$, we have

$$I(\lambda; -a) = -e^{i\lambda a^2} \int_0^\infty e^{2\lambda x(y) y} e^{\sin(x(y) - iy)} [x'(y) + i]\, \mathrm{d}y$$

Using Laplace's method, we have

$$R_{\max} = R(0) = 0, \qquad g(0) = ie^{-\sin a}, \qquad R'(0) = 2a$$

The asymptotic expansion is

$$I(\lambda; -a) \sim -ie^{i\lambda a^2} \frac{e^{-\sin a}}{2\lambda a} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty$$

Similarly, for $x_0 = a > 0$ we have

$$I(\lambda; a) \sim ie^{i\lambda a^2} \frac{e^{\sin a}}{2\lambda a} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to +\infty$$

When $x_0 = 0$, the saddle point needs to be analyzed. We consider the steepest descent path as the whole diagonal which gives

$$I_s(\lambda; 0) = \int_{-\infty}^{+\infty} e^{-2\lambda y^2} e^{\sin(1+i)y}\, (1+i)\, \mathrm{d}y$$

Using Laplace's method, we have

$$R_{\max} = R(0) = 0, \qquad g(0) = 1 + i, \qquad R''(0) = -4$$

The asymptotic expansion is

$$I_s(\lambda; 0) \sim (1+i)\sqrt{\frac{\pi}{2\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right), \qquad \lambda \to +\infty$$

Now we can summarize the leading order asymptotes for integrals over different intervals.

$$[a, b] = [-2\pi, -\pi], \qquad I \sim \frac{ie^{i\lambda\pi^2}}{2\pi\lambda} - \frac{ie^{i\lambda 4\pi^2}}{4\pi\lambda} + o(\lambda^{-1})$$

$$[a, b] = [-\pi, \pi], \qquad I \sim (1+i)\sqrt{\frac{\pi}{2\lambda}} + o\left(\lambda^{-1/2}\right)$$

$$[a, b] = [\pi, 2\pi], \qquad I \sim \frac{ie^{i\lambda\pi^2}}{2\pi\lambda} - \frac{ie^{i\lambda 4\pi^2}}{4\pi\lambda} + o(\lambda^{-1})$$

The results match with the analysis from the stationary phase method.

Degenerate stationary phase point

$$F_\delta(\lambda) = \int_{t_0 - 2\delta}^{t_0 + 2\delta} e^{i\lambda I(t)}\, \mathrm{d}t\,, \qquad \lambda \to +\infty$$

Now we consider a stationary phase point t_0 with $I'(t_0) = I''(t_0) = 0$ but $I'''(t_0) > 0$. From previous analysis, around $t = t_0$, we first write the Taylor series of $I(t)$

$$I(t) = I(t_0) + \frac{1}{6} I'''(t_0)(t - t_0)^3 + \cdots$$

Denote $\sigma = \operatorname{sgn} I'''(t_0)$. According to Morse Lemma, we find a transformation such that

$$I(t) = I(t_0) + \sigma s^3, \qquad F_\delta(\lambda) = e^{i\lambda I(t_0)} \int_{s_-}^{s_+} e^{i\lambda\sigma s^3} t'(s)\, \mathrm{d}s$$

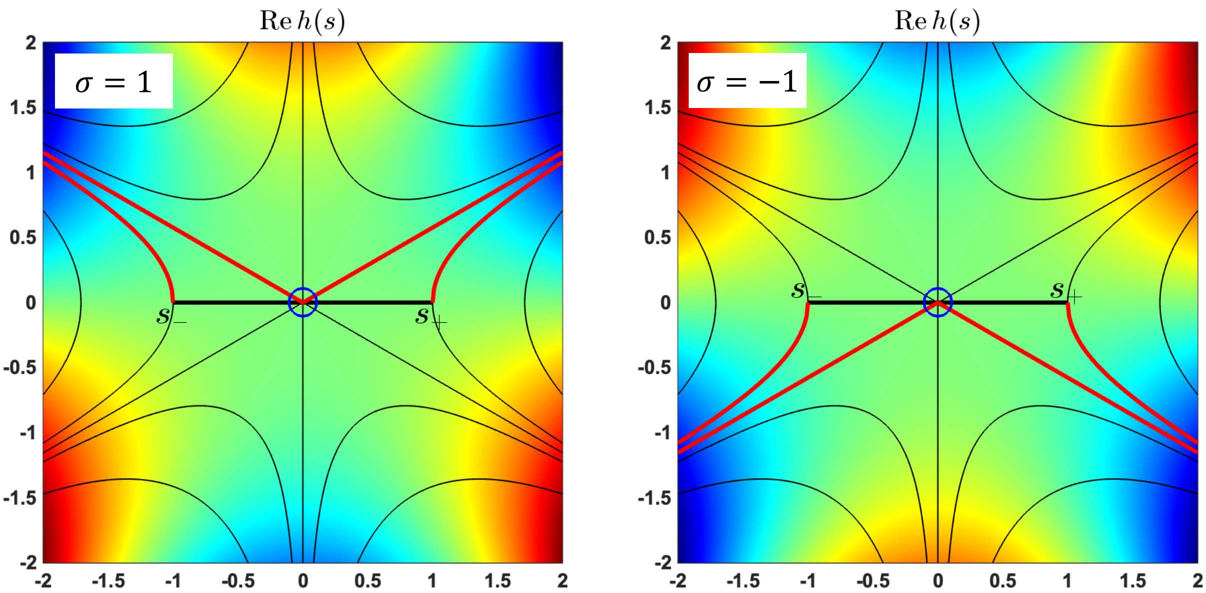
The lower and upper limits of the integral are calculated as

$$s_\pm = \sqrt[3]{\sigma[I(t_0 \pm 2\delta) - I(t_0)]}, \qquad t(s), s(t) \in C^{2N+1}, \qquad t'(s) \in C^{2N}$$

We can similarly analyze F_δ using the method of steepest descent. For $\sigma = 1$, we have

$$h(s) = is^3, \qquad u = y^3 - 3x^2 y, \qquad v = x^3 - 3xy^2$$

The steepest descent path is constructed as shown in the figures. Different from the case for z^2,



now the path changes direction at the saddle point $s = 0$ to ensure the real part having $u < 0$. We only need to evaluate the integral along the diagonals, and note that

$$\int_{\gamma_1 \cup \gamma_2} e^{i\lambda s^3} \mathrm{d}s = \left(-e^{\frac{5i\pi}{6}} + e^{\frac{i\pi}{6}}\right) \int_0^{+\infty} e^{-\lambda u^3} \mathrm{d}u = \sqrt{3}\Gamma\left(\frac{4}{3}\right) \lambda^{-\frac{1}{3}}$$

Now we obtain

$$\int_{s_-}^{s_+} e^{i\lambda\sigma s^3} t'(s)\, \mathrm{d}s \sim \sqrt{3}\Gamma\left(\frac{4}{3}\right)\lambda^{-\frac{1}{3}} t'(0) + O\left(\lambda^{-\frac{2}{3}}\right), \qquad \lambda \to +\infty$$

By taking derivatives of the transformation, we have

$$t'(0) = \left(\frac{6}{|I^{(3)}(t_0)|}\right)^{\frac{1}{3}}$$

The final asymptotic expansion becomes

$$F_\delta(\lambda) \sim \sqrt{3}\Gamma\left(\frac{4}{3}\right) e^{i\lambda I(t_0)} \left(\frac{6}{\lambda |I^{(3)}(t_0)|}\right)^{\frac{1}{3}} + O\left(\lambda^{-\frac{2}{3}}\right), \qquad \lambda \to +\infty$$

Oscillatory integral

$$F(\lambda) = \int_{-\pi/2}^{\pi/2} \cos(nt - \lambda \cos t)\ \mathrm{d}t = \mathrm{Re} \int_{-\pi/2}^{\pi/2} e^{i(nt - \lambda\cos t)}\, \mathrm{d}t\,, \qquad \lambda \to +\infty$$

We recognize

$$I(t) = -\cos t\,, \qquad g(t) = e^{int}, \qquad t_0 = 0, \qquad I(t_0) = -1, \qquad I''(t_0) = 1$$

The leading order term does not depend on n, since the stationary phase point is $t_0 = 0$.

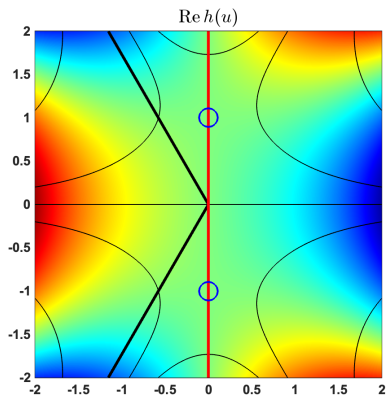
$$F_n(\lambda) \sim \mathrm{Re}\left[e^{-i\lambda} e^{\frac{i\pi}{4}} \sqrt{2\pi/\lambda}\right] \sim \sqrt{2\pi/\lambda} \cos\left(\lambda - \frac{\pi}{4}\right) + O(\lambda^{-1}), \qquad \lambda \to +\infty$$

Airy function

The asymptotic behavior of $\mathrm{Ai}(x)$ as $x \to -\infty$ can also be obtained by the method of stationary phase. Denote $r = -x > 0$ and $\lambda = r^{3/2}$, we have

$$\mathrm{Ai}(-r) = \frac{\sqrt{r}}{2\pi i}\int_C e^{\lambda h(u)}\,\mathrm{d}u\,, \qquad h(u) = -u - \frac{u^3}{3}, \qquad \lambda \to +\infty$$

The path can be deformed to the imaginary axis. This is because the integrand is analytic, and the integral over the two arcs tends to zero as $R \to \infty$ since the real part is negative. Therefore, we have



$$\mathrm{Ai}(-r) = \frac{\sqrt{r}}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda I(t)}\mathrm{d}t\,, \qquad I(t) = -t + \frac{t^3}{3}$$

There are two stationary phase points in the interval

$$t_0 = \pm 1, \qquad I(t_0) = \mp\frac{2}{3}, \qquad I''(t_0) = \pm 2$$

The leading order terms are obtained as

$$\int_{-\infty}^{+\infty} e^{i\lambda I(t)}\mathrm{d}t = e^{-\frac{2i\lambda}{3}}e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{\lambda}} + e^{\frac{2i\lambda}{3}}e^{-\frac{i\pi}{4}}\sqrt{\frac{\pi}{\lambda}} + O\left(\lambda^{-3/2}\right), \qquad \lambda \to +\infty$$

Since the contribution from endpoints at infinity is zero, the next order term is modified accordingly. Finally, we have the asymptotic expansion for $\mathrm{Ai}(x)$ as

$$\mathrm{Ai}(x) = \frac{|x|^{-\frac{1}{4}}}{\sqrt{\pi}}\cos\left(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{\pi}{4}\right) + O\left(|x|^{-\frac{7}{4}}\right), \qquad x \to -\infty$$