MIT Integration Bee: 2026 Regular Season

Question 1

$$\int_0^{2026} \left(\sum_{k=0}^{\infty} \frac{x^{2026k+2025}}{k!} \right) \mathrm{d}x \tag{1.1}$$

Solution Based on the **Taylor series** of e^x, we can recognize that

$$I = \int_0^{2026} x^{2025}\, e^{x^{2026}}\, \mathrm{d}x = \frac{e^{2026^{2026}} - 1}{2026}. \tag{1.2}$$

Question 2

$$\int_0^{\ln 34} \left(2 + \frac{1}{2e^x - 1} + \frac{1}{3e^x - 1} \right) \mathrm{d}x \tag{2.1}$$

Solution

$$I = \int_0^{\ln 34} \left(\frac{2e^x}{2e^x - 1} + \frac{3e^x}{3e^x - 1} \right) \mathrm{d}x = \ln 67 + \ln 101 - \ln 2 = \ln\left(\frac{6767}{2}\right). \tag{2.2}$$

Question 3

$$\int_0^{1/2} \left[\cos(\pi x) - \pi \left(\frac{1}{4} - x^2 \right) \left(\frac{5}{4} - x^2 \right) \right] \mathrm{d}x = \frac{1}{\pi} - \frac{\pi}{10}. \tag{3.1}$$

Question 4

$$\int_{\int_{\int_0^1 x\,\mathrm{d}x}^{1} x\,\mathrm{d}x}^{\int_0^{\int_0^{\int_0^1 x\,\mathrm{d}x} x\,\mathrm{d}x} x\,\mathrm{d}x} x\,\mathrm{d}x = \int_{55/128}^{1/128} x\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{128^2} - \frac{55^2}{128^2} \right) = -\frac{189}{2048}. \tag{4.1}$$

Question 5

$$\int \frac{4x^2-1}{x^2\left(x^2-1\right)}\,\mathrm{d}x = \int \left(\frac{1}{x^2}+\frac{3}{x^2-1}\right)\mathrm{d}x = -\frac{1}{x}+\frac{3}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \tag{5.1}$$

Question 6

$$\int \frac{\mathrm{d}x}{27x-x^{-1/3}} \tag{6.1}$$

Solution With the following **change of variable**

$$t = x^{-1/3}, \qquad x = t^{-3}, \qquad \mathrm{d}x = -3t^{-4}\,\mathrm{d}t, \tag{6.2}$$

the integral becomes

$$I = \int \frac{-3t^{-5}}{27t^{-4}-1}\,\mathrm{d}t = \frac{1}{36}\ln\left|27t^{-4}-1\right|+C = \frac{1}{36}\ln\left|27x^{4/3}-1\right|+C. \tag{6.3}$$

Question 7

$$\int \frac{2\sqrt{x}+1}{\sqrt{x^2+x\sqrt{x}}}\,\mathrm{d}x \tag{7.1}$$

Solution Note that

$$\left(\sqrt{x+\sqrt{x}}\right)' = \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right) = \frac{2\sqrt{x}+1}{4\sqrt{x^2+x\sqrt{x}}}, \qquad I = 4\sqrt{x+\sqrt{x}}+C. \tag{7.2}$$

Question 8

$$\int \frac{e^x}{e^{2x}-e^{-2x}}\,\mathrm{d}x \tag{8.1}$$

Solution With the **change of variable** $t = e^x$, we have

$$I = \int \frac{t^2}{t^4-1}\,\mathrm{d}t = \frac{1}{2}\arctan e^x + \frac{1}{4}\ln\left|\frac{1-e^x}{1+e^x}\right|+C. \tag{8.2}$$

Question 9

$$\int_0^\pi \frac{|\sin x + \sin 2x + \sin 3x + \sin 4x|}{\sin x + \sin 2x + \sin 3x + \sin 4x}\,\mathrm{d}x \tag{9.1}$$

Solution Note that

$$f(x) = \sin x + \sin 2x + \sin 3x + \sin 4x = 4\sin\left(\frac{5x}{2}\right)\cos x\cos\left(\frac{x}{2}\right) \tag{9.2}$$

Based on the sign of $f(x)$, the integral is thus evaluated as

$$I = \frac{2\pi}{5} - \frac{\pi}{10} + \frac{3\pi}{10} - \frac{\pi}{5} = \frac{2\pi}{5}. \tag{9.3}$$

Question 10

$$\int_0^1 \frac{1-x}{\sqrt{e^{2x}-x^2}}\,\mathrm{d}x. \tag{10.1}$$

Solution

$$I = \int_0^1 \frac{e^{-x}(1-x)}{\sqrt{1-x^2e^{-2x}}}\,\mathrm{d}x = \left[\arcsin\left(xe^{-x}\right)\right]_0^1 = \arcsin\left(\frac{1}{e}\right) \tag{10.2}$$

Question 11

$$\int_0^{2026} \left\{x + \frac{1}{2}\left\lfloor\frac{x}{2}\right\rfloor + \frac{1}{3}\left\lfloor\frac{x}{3}\right\rfloor + \frac{1}{4}\left\lfloor\frac{x}{4}\right\rfloor + \cdots\right\}\mathrm{d}x \tag{11.1}$$

Solution For $x \in [n, n+1)$ with integer $n \geq 0$, we have

$$\frac{1}{2}\left\lfloor\frac{x}{2}\right\rfloor + \frac{1}{3}\left\lfloor\frac{x}{3}\right\rfloor + \frac{1}{4}\left\lfloor\frac{x}{4}\right\rfloor + \cdots = \text{const.} \tag{11.2}$$

Note that this constant does not influence the integral of $\{x\}$. Therefore, we obtain

$$I = \sum_{k=0}^{2025}\int_0^1 \{x + c_k\}\,\mathrm{d}x = 2026\int_0^1 \{x\}\,\mathrm{d}x = 1013. \tag{11.3}$$

Question 12

$$\int_{1/2}^{2} \frac{x^8}{x^8 - x^6 + x^4 - x^2 + 1}\, dx \tag{12.1}$$

Solution Note that

$$\left(x^2 + 1\right)\left(x^8 - x^6 + x^4 - x^2 + 1\right) = x^{10} + 1. \tag{12.2}$$

The symmetry of the limits of integration inspires the following **change of variable**

$$t = \frac{1}{x}, \qquad I = \int_{1/2}^{2} \frac{dt}{t^2\left(t^8 - t^6 + t^4 - t^2 + 1\right)} \tag{12.3}$$

Now we have

$$2I = \int_{1/2}^{2} \frac{x^{10} + 1}{x^2\left(x^8 - x^6 + x^4 - x^2 + 1\right)}\, dx = \int_{1/2}^{2} \left(1 + \frac{1}{x^2}\right) dx = 3, \qquad I = \frac{3}{2}. \tag{12.4}$$

Question 13

$$\int \operatorname{arcsinh}^2 x \, dx \tag{13.1}$$

Solution Repeatedly using **integration by parts**, we have

$$\begin{aligned} I &= x \operatorname{arcsinh}^2 x - 2 \int \frac{x \operatorname{arcsinh} x}{\sqrt{x^2 + 1}}\, dx = x \operatorname{arcsinh}^2 x - 2 \int \operatorname{arcsinh} x \, d\left(\sqrt{x^2 + 1}\right) \\ &= x \operatorname{arcsinh}^2 x - 2\sqrt{x^2 + 1}\operatorname{arcsinh} x + 2x + C. \end{aligned} \tag{13.2}$$

Question 14

$$\int \cos^2(2026x)\cos(1013x)\, dx \tag{14.1}$$

Solution Note that

$$\cos^2(2\alpha)\cos\alpha = \frac{\cos\alpha}{2} + \frac{\cos(4\alpha)\cos\alpha}{2} = \frac{\cos\alpha}{2} + \frac{\cos 3\alpha}{4} + \frac{\cos 5\alpha}{4}. \tag{14.2}$$

The integral is thus solved as

$$I = \frac{\sin(1013x)}{2026} + \frac{\sin(3039x)}{12156} + \frac{\sin(5065x)}{20260} + C. \tag{14.3}$$

Question 15

$$\int e^x \arcsin(\tanh x)\,\mathrm{d}x \tag{15.1}$$

Solution Using **integration by parts**, we have

$$\begin{aligned}
I &= e^x \arcsin(\tanh x) - \int \frac{e^x}{\sqrt{1-\tanh^2 x}\cosh^2 x}\,\mathrm{d}x \\
&= e^x \arcsin(\tanh x) - \int \frac{e^x\,\mathrm{d}x}{\cosh x} = e^x \arcsin(\tanh x) - \int \frac{2e^{2x}}{e^{2x}+1}\,\mathrm{d}x \\
&= e^x \arcsin(\tanh x) - \ln\left(e^{2x}+1\right) + C.
\end{aligned} \tag{15.2}$$

Question 16

$$\int \frac{x\left[\left(1-x^2\right)\cos x + 2x\sin x\right]}{\left(1+x^2\right)^2}\,\mathrm{d}x \tag{16.1}$$

Solution Denote the following functions

$$u = \left(1-x^2\right)\cos x + 2x\sin x, \qquad v = -\frac{1}{2\left(1+x^2\right)}. \tag{16.2}$$

Their derivatives are

$$u' = \left(x^2+1\right)\sin x, \qquad v' = \frac{x}{\left(1+x^2\right)^2}. \tag{16.3}$$

Using **integration by parts**, we have

$$\begin{aligned}
I &= \int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x \\
&= -\frac{\left(1-x^2\right)\cos x + 2x\sin x}{2\left(1+x^2\right)} + \frac{1}{2}\int \sin x\,\mathrm{d}x = -\frac{\cos x + x\sin x}{1+x^2} + C.
\end{aligned} \tag{16.4}$$

Question 17

$$\int_0^1 \sin\left[(x-1)(5x-1)\right] \sin\left[x(3x-2)\right] \mathrm{d}x \tag{17.1}$$

Solution

$$\begin{aligned} I &= \frac{1}{2}\int_0^1 \cos\left(2x^2-4x+1\right)\mathrm{d}x - \frac{1}{2}\int_0^1 \cos\left(8x^2-8x+1\right)\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \cos\left(2x^2-4x+1\right)\mathrm{d}x - \int_0^{1/2} \cos\left(8x^2-8x+1\right)\mathrm{d}x. \end{aligned} \tag{17.2}$$

The arguments are now monotonic over the interval

$$u(x) = 2x^2 - 4x + 1, \qquad v(x) = 8x^2 - 8x + 1, \tag{17.3}$$

with the differentials calculated as

$$\mathrm{d}x = \frac{\mathrm{d}u}{2\sqrt{2}\sqrt{u+1}}, \qquad \mathrm{d}x = \frac{\mathrm{d}v}{4\sqrt{2}\sqrt{v+1}}. \tag{17.4}$$

We note that

$$I = \frac{1}{4\sqrt{2}}\int_1^{-1} \frac{\mathrm{d}u}{\sqrt{u+1}} - \frac{1}{4\sqrt{2}}\int_1^{-1} \frac{\mathrm{d}v}{\sqrt{v+1}} = 0. \tag{17.5}$$

Question 18

$$\int_{1/4}^{\sqrt{2}} \frac{\mathrm{d}x}{x\left(x^{x^{x^{\cdots}}}\ln x - 1\right)} \tag{18.1}$$

Solution With the following **change of variable**

$$t = x^{x^{x^{\cdots}}}, \qquad \ln t = t\ln x, \qquad \frac{\mathrm{d}x}{x} = \frac{1-\ln t}{t^2}\,\mathrm{d}t, \tag{18.2}$$

the limits of integration becomes $[t_1, t_2]$ where t_1 and t_2 satisfy

$$\frac{\ln t_1}{t_1} = \ln\frac{1}{4} = 2\ln\frac{1}{2}, \qquad \frac{\ln t_2}{t_2} = \ln\sqrt{2} = \frac{\ln 2}{2}. \tag{18.3}$$

We can observe that $t_1 = 1/2$ and $t_2 = 2$. The integral is thus evaluated as

$$I = \int_{1/2}^{2} \frac{1}{t\ln x - 1}\cdot\frac{1-\ln t}{t^2}\,\mathrm{d}t = -\int_{1/2}^{2}\frac{\mathrm{d}t}{t^2} = -\frac{3}{2}. \tag{18.4}$$

Question 19

$$\int_0^1 \frac{1+\left[\left(2026+2x-x^2\right)^2-2\right]\left(2026+4x-4x^2\right)^2}{\left(2025+2x-x^2\right)\left(2027+2x-x^2\right)\left(2025+4x-4x^2\right)\left(2027+4x-4x^2\right)}\,\mathrm{d}x \tag{19.1}$$

Solution Denote the following function

$$u(x)=2026+2x-x^2, \qquad v(x)=2026+4x-4x^2. \tag{19.2}$$

The integrand becomes

$$f(x)=\frac{1+\left(u^2-2\right)v^2}{\left(u^2-1\right)\left(v^2-1\right)}=1+\frac{u^2-v^2}{\left(u^2-1\right)\left(v^2-1\right)}=1+\frac{1}{v^2-1}-\frac{1}{u^2-1}. \tag{19.3}$$

Note that $v(x)=u(2x)$ and $u(x)=u(2-x)$. Therefore, we have

$$\int_0^1 \frac{\mathrm{d}x}{v^2(x)-1}=\int_0^1 \frac{\mathrm{d}x}{u^2(2x)-1}=\frac{1}{2}\int_0^2 \frac{\mathrm{d}t}{u^2(t)-1}=\int_0^1 \frac{\mathrm{d}t}{u^2(t)-1}. \tag{19.4}$$

Finally, the original integral is evaluated as

$$I=\int_0^1 f(x)\,\mathrm{d}x=1. \tag{19.5}$$

Question 20

$$\int_{1/2}^{1} 4^{x-1}\left(1+4^{4^{x-1}-1}\left(1+4^{4^{4^{x-1}-1}-1}\left(1+\cdots\right)\right)\right)\,\mathrm{d}x \tag{20.1}$$

Solution Define the following function series

$$a_0(x)=x,\qquad a_1(x)=4^{x-1}=4^{a_0-1},\qquad a_k(x)=4^{a_{k-1}-1},\qquad k\in\mathbb{N}^*. \tag{20.2}$$

The differentials can be obtained as

$$\mathrm{d}a_0=\mathrm{d}x,\qquad \mathrm{d}a_1=\ln 4\,a_1\,\mathrm{d}a_0,\qquad \mathrm{d}a_k=\ln 4\,a_k\,\mathrm{d}a_{k-1},\qquad k\in\mathbb{N}^*. \tag{20.3}$$

Also note that the limits of integration are the fixed points of the series

$$a_k\left(\frac{1}{2}\right)=\frac{1}{2},\qquad a_k(1)=1,\qquad k\in\mathbb{N}. \tag{20.4}$$

Now we can write the integral as

$$I=\int_{1/2}^{1}\left(a_1+a_1a_2+a_1a_2a_3+\cdots\right)\,\mathrm{d}x. \tag{20.5}$$

For $k\geq 1$, each term can be evaluated as

$$I_k=\int_{1/2}^{1}\prod_{i=1}^{k}a_i\,\mathrm{d}x=\frac{1}{\ln 4}\int_{1/2}^{1}\prod_{i=2}^{k}a_i\,\mathrm{d}a_1=\cdots=\frac{1}{(\ln 4)^{k-1}}\int_{1/2}^{1}a_k\,\mathrm{d}a_{k-1}=\frac{I_1}{(\ln 4)^{k-1}}. \tag{20.6}$$

Eventually, the integral is the sum of a geometric series

$$I_1=\frac{1}{2\ln 4},\qquad I=\sum_{k=1}^{\infty}I_k=\frac{I_1}{1-(\ln 4)^{-1}}=\frac{1}{2\left(\ln 4-1\right)}. \tag{20.7}$$