MIT Integration Bee: 2026 Final

Question 1

$$\lim_{n\to\infty}\int_0^1\left(\sum_{k=1}^{n}\frac{1}{nx^2+kx+n}\right)\mathrm{d}x. \tag{1.1}$$

Solution The summation can be converted into a **Riemann sum**, which can be further recognized as an integral

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x^2+\frac{k}{n}x+1}=\int_0^1\frac{\mathrm{d}t}{x^2+tx+1}=\frac{\ln\left(x^2+x+1\right)-\ln\left(x^2+1\right)}{x}. \tag{1.2}$$

For the second term, we have (see 2024 Quarterfinal #2: Question 2)

$$I_2=\int_0^1\frac{\ln\left(1+x^2\right)}{x}\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{2n-1}\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n^2}=\frac{\pi^2}{24}. \tag{1.3}$$

The infinite sum is from the **Basel problem** (see 2024 Final: Question 2). For the first term, we can write it as

$$I_1=\int_0^1\frac{\ln\left(1+x+x^2\right)}{x}\,\mathrm{d}x=\int_0^1\frac{\ln\left(1-\omega x\right)}{x}\,\mathrm{d}x+\int_0^1\frac{\ln\left(1-\omega^2 x\right)}{x}\,\mathrm{d}x, \tag{1.4}$$

where $\omega=e^{2\pi i/3}$ is the **root of unity**. Similarly, based on the following result

$$\int_0^1\frac{\ln\left(1-\alpha x\right)}{x}\,\mathrm{d}x=-\sum_{n=1}^{\infty}\frac{\alpha^n}{n}\int_0^1 x^{n-1}\,\mathrm{d}x=-\frac{\alpha^n}{n^2}, \tag{1.5}$$

we have

$$I_1=-\sum_{n=1}^{\infty}\frac{\omega^n+\omega^{2n}}{n^2}=-2\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^2},\qquad\text{with }\theta=\frac{2\pi}{3}. \tag{1.6}$$

Using the following **Fourier series**

$$(x-\pi)^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2},\qquad x\in[0,\pi], \tag{1.7}$$

we can obtain

$$F(\theta)=\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^2}=\frac{\pi^2}{6}-\frac{\pi\theta}{2}+\frac{\theta^2}{4},\qquad\theta\in[0,\pi]. \tag{1.8}$$

Finally, the original problem is solved as

$$I=I_1-I_2=-2F\left(\frac{2\pi}{3}\right)-\frac{\pi^2}{24}=\frac{\pi^2}{9}-\frac{\pi^2}{24}=\frac{5\pi^2}{72}. \tag{1.9}$$

Question 2

$$\int \frac{\mathrm{d}x}{(x-1)\sqrt[4]{x^3+x}} \tag{2.1}$$

Solution Inspired by the **Euler substitution**, we use the following **change of variable**

$$t = \frac{\sqrt[4]{x^3+x}}{x+1}, \qquad \frac{\mathrm{d}t}{t} = \left[\frac{3x^2+1}{4x\left(x^2+1\right)} - \frac{1}{x+1}\right]\mathrm{d}x = -\frac{(x-1)^3}{4x\,(x+1)\left(x^2+1\right)}\,\mathrm{d}x. \tag{2.2}$$

We also obtain

$$x^3 + x = t^4\,(x+1)^4\,, \qquad \left(\frac{x-1}{x+1}\right)^4 = 1 - \frac{8\left(x^3+x\right)}{(x+1)^4} = 1 - 8t^4. \tag{2.3}$$

The integral now becomes

$$\begin{aligned} I &= -\int \frac{4\left(x^3+x\right)}{t^2\,(x-1)^4}\,\mathrm{d}t = -4\int t^2\left(\frac{x+1}{x-1}\right)^4\mathrm{d}t = -4\int \frac{t^2}{1-8t^4}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2}}\left(\int \frac{\mathrm{d}t}{2\sqrt{2}t^2+1} + \int \frac{\mathrm{d}t}{2\sqrt{2}t^2-1}\right). \end{aligned} \tag{2.4}$$

With the following results

$$\int \frac{\mathrm{d}t}{1+at^2} = \frac{1}{\sqrt{a}}\arctan\left(\sqrt{a}t\right) + C, \qquad \int \frac{\mathrm{d}t}{1-at^2} = \frac{1}{\sqrt{a}}\operatorname{arctanh}\left(\sqrt{a}t\right) + C, \tag{2.5}$$

the original problem is solved as

$$I = \frac{1}{2\sqrt[4]{2}}\left[\arctan\left(\frac{\sqrt[4]{8x^3+8x}}{x+1}\right) - \operatorname{arctanh}\left(\frac{\sqrt[4]{8x^3+8x}}{x+1}\right)\right] + C. \tag{2.6}$$

Question 3

$$\lim_{n\to\infty}\int_0^{2026}\underbrace{\log_{\sqrt{2}}\left(x+\log_{\sqrt{2}}\left(x+\cdots+\log_{\sqrt{2}}(x+2026)\right)\right)}_{n\,\text{logs}}\,\mathrm{d}x \tag{3.1}$$

Solution Define the following function series

$$a_0(x)=2026,\qquad a_1(x)=\log_{\sqrt{2}}(x+a_0),\qquad a_k(x)=\log_{\sqrt{2}}(x+a_{k-1}),\qquad k\in\mathbb{N}^*. \tag{3.2}$$

As $k\to\infty$, denote $a_k(x)\to f(x)$ and we have

$$y=f(x)=\log_{\sqrt{2}}(x+f(x)),\qquad x=t(y)=2^{y/2}-y. \tag{3.3}$$

It can be shown that the series converges within the interval. Note that

$$t'(y)=2^{y/2-1}\ln 2-1>0,\qquad \forall y\geq 4;\qquad t(4)=0,\qquad t(22)=2026. \tag{3.4}$$

Therefore, using **integration by parts**, the integral is evaluated as

$$\begin{aligned} I&=\int_0^{2026}f(x)\,\mathrm{d}x=[xf(x)]_0^{2026}-\int_4^{22}t(y)\,\mathrm{d}y\\ &=2026\times 22-\left[\frac{2^{y/2+1}}{\ln 2}-\frac{1}{2}y^2\right]_4^{22}=44806-\frac{4088}{\ln 2}. \end{aligned} \tag{3.5}$$

Question 4

$$\int_0^{1/4} \left(\left\lfloor \sqrt[4]{\frac{1}{x} - 4} \right\rfloor^2 + \left\lfloor \sqrt[4]{\frac{1}{x} - 4} \right\rfloor \right) \mathrm{d}x \tag{4.1}$$

Solution Note that

$$\left\lfloor \sqrt[4]{\frac{1}{x} - 4} \right\rfloor = k, \qquad \text{when} \quad \frac{1}{(k+1)^4 + 4} < x \leq \frac{1}{k^4 + 4}. \tag{4.2}$$

The integral becomes the infinite sum

$$I = \sum_{k=0}^{\infty} k\,(k+1) \left[\frac{1}{k^4 + 4} - \frac{1}{(k+1)^4 + 4} \right] = \sum_{k=0}^{\infty} a_k \,(b_k - b_{k+1})\,, \tag{4.3}$$

where the two series $\{a_k\}$ and $\{b_k\}$ are defined as

$$a_k = k\,(k+1)\,, \qquad b_k = \frac{1}{k^4 + 4} \qquad k \in \mathbb{N}. \tag{4.4}$$

Summation by parts leads to

$$I = a_0 b_0 + \sum_{k=1}^{\infty} (a_k - a_{k-1})\, b_k = \sum_{k=1}^{\infty} \frac{2k}{k^4 + 4}. \tag{4.5}$$

This can be transformed into a **telescoping series** as

$$\frac{4k}{k^4 + 4} = \frac{1}{(k-1)^2 + 1} - \frac{1}{(k+1)^2 + 1}. \tag{4.6}$$

Therefore, the final result is obtained as

$$I = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}. \tag{4.7}$$

Question 5

$$\int_{-\infty}^{+\infty} \left(\left(\frac{1}{x-2} + \frac{3}{x-4} + \frac{5}{x-6} \right)^{-2} + 1 \right)^{-1} \mathrm{d}x \tag{5.1}$$

Solution Denote the partial fraction as $f(x)$, and then we have

$$I = \int_{-\infty}^{+\infty} \frac{f^2(x)}{1+f^2(x)} \mathrm{d}x, \qquad f(x) = \frac{1}{x-2} + \frac{3}{x-4} + \frac{5}{x-6} = \sum_{j=1}^{3} \frac{a_j}{x-m_j}. \tag{5.2}$$

Consider the following rational function in the complex plane

$$R(z) = \frac{f(z)}{1+if(z)} = \frac{f(z) - if^2(z)}{1+f^2(z)}. \tag{5.3}$$

First, the poles of $R(z)$ are all in the lower half-plane. This is because with $z = x + iy$, we have

$$1 + if(z) = 0 \quad \Leftrightarrow \quad f(z) = \sum_{j=1}^{3} \frac{(x-m_j)\, a_j - iya_j}{(x-m_j)^2 + y^2} = i. \tag{5.4}$$

To satisfy this equation, we require $y < 0$ in the lower half-plane. From the **residue theorem**, the integral over a contour in the upper half-plane is zero for the rational function. As $z \to \infty$, we have

$$f(z) \to \frac{9}{z} + O\left(\frac{1}{z^2}\right), \qquad R(z) \to \frac{9}{z} + O\left(\frac{1}{z^2}\right). \tag{5.5}$$

Therefore, we have

$$\lim_{R\to\infty} I_R = \lim_{R\to\infty} \int_{C_R} R(z)\, \mathrm{d}z = 9\pi i. \tag{5.6}$$

This leads to

$$\int_{-\infty}^{+\infty} R(z)\, \mathrm{d}z + \lim_{R\to\infty} I_R = 0, \qquad \int_{-\infty}^{+\infty} R(z)\, \mathrm{d}z = -9\pi i. \tag{5.7}$$

The original problem is thus evaluated as

$$I = -\mathrm{Im} \int_{-\infty}^{+\infty} R(z)\, \mathrm{d}z = 9\pi. \tag{5.8}$$