MIT Integration Bee: 2025 Semifinal

Semifinal #1

Question 1

$$\int_0^{+\infty} \frac{\sqrt[3]{x}}{1+x^2}\,\mathrm{d}x \tag{1.1}$$

Solution With the following **change of variable**

$$t = \sqrt[3]{x}, \qquad x = t^3, \qquad \mathrm{d}x = 3t^2\,\mathrm{d}t, \tag{1.2}$$

the integral becomes

$$I = 3\int_0^{+\infty} \frac{t^3\,\mathrm{d}t}{1+t^6}. \tag{1.3}$$

We consider a more general integral

$$I(m,n) = \int_0^{+\infty} \frac{x^m\,\mathrm{d}x}{1+x^n}, \qquad \text{with } m < n. \tag{1.4}$$

Using the **residue theorem** on the contour in Fig. 1, we have

$$\left(1 - e^{i(m+1)\theta}\right) I(m,n) = 2\pi i \cdot \mathrm{Res}\left(f, z_0 = e^{i\theta/2}\right) = 2\pi i \cdot \frac{z_0^m}{n\, z_0^{n-1}}. \tag{1.5}$$

After some algebra, we have

$$I(m,n) = \frac{\pi}{n}\cdot\frac{1}{\sin\left(\frac{m+1}{n}\pi\right)}, \qquad I = 3\cdot I(3,6) = \frac{\pi}{\sqrt{3}}. \tag{1.6}$$

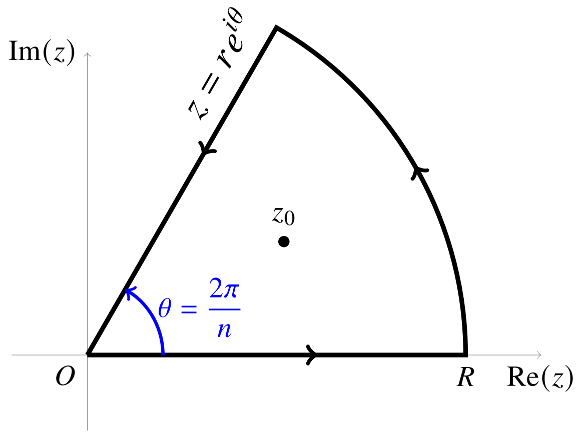

Fig. 1 Sector-shaped contour in the upper plane to evaluate the integral.

Question 2

$$\int_{-\pi}^{\pi} \ln\left[82 + 2\left(\cos x \cdot \sqrt{81 - \sin^2 x} - \sin^2 x\right)\right] \mathrm{d}x \tag{2.1}$$

Solution Note that

$$\begin{aligned} 82 + 2\left(\cos x \cdot \sqrt{81 - \sin^2 x} - \sin^2 x\right) &= 80 + 2\cos^2 x + 2\cos x\sqrt{80 + \cos^2 x} \\ &= \left(\sqrt{80 + \cos^2 x} + \cos x\right)^2 . \end{aligned} \tag{2.2}$$

In addition, taking advantage of the following pairs

$$\left(\sqrt{80 + \cos^2 x} + \cos x\right)\left(\sqrt{80 + \cos^2 x} - \cos x\right) = 80, \tag{2.3}$$

with a **change of variable** $u = \pi - x$, we obtain

$$\int_0^{\pi} \ln\left(\sqrt{80 + \cos^2 x} + \cos x\right) \mathrm{d}x = \int_0^{\pi} \ln\left(\sqrt{80 + \cos^2 u} - \cos u\right) \mathrm{d}u. \tag{2.4}$$

Finally, the original integral is evaluated as

$$\begin{aligned} I &= 2\int_{-\pi}^{\pi} \ln\left(\sqrt{80 + \cos^2 x} + \cos x\right) \mathrm{d}x = 4\int_0^{\pi} \ln\left(\sqrt{80 + \cos^2 x} + \cos x\right) \mathrm{d}x \\ &= 2\int_0^{\pi} \ln\left(\sqrt{80 + \cos^2 x} + \cos x\right) \mathrm{d}x + 2\int_0^{\pi} \ln\left(\sqrt{80 + \cos^2 x} - \cos x\right) \mathrm{d}x \\ &= 2\int_0^{\pi} \ln 80 \,\mathrm{d}x = 2\pi \ln 80. \end{aligned} \tag{2.5}$$

Question 3

$$\int \left(3x^2 + 7x - 5\right) \left(x + \frac{1}{x}\right) e^{x+\frac{1}{x}} \, \mathrm{d}x \tag{3.1}$$

Solution We expect the result to have the following form

$$F(x) = f(x)\, e^{x+\frac{1}{x}}, \qquad F'(x) = \left(3x^2 + 7x - 5\right) \left(x + \frac{1}{x}\right) e^{x+\frac{1}{x}}. \tag{3.2}$$

Therefore, the polynomial $f(x)$ satisfies

$$f(x) + f'(x) - \frac{f(x)}{x^2} = 3x^3 + 7x^2 - 2x + 7 - \frac{5}{x}. \tag{3.3}$$

The leading term of $f(x)$ must be $3x^3$, and we can subtract the contributions from this term on both sides. Keeping doing this, and we finally obtain

$$I = \left(3x^3 - 2x^2 + 5x\right) e^{x+\frac{1}{x}} + C. \tag{3.4}$$

Question 4

$$\int_0^{+\infty} \frac{x}{e^{2x} + 1} \, \mathrm{d}x \tag{4.1}$$

Solution Based on **Taylor expansion**, we have

$$\begin{aligned} I &= \int_0^{+\infty} \frac{x}{e^{2x}} \cdot \frac{1}{1 + e^{-2x}} \, \mathrm{d}x = \sum_{n=0}^{+\infty} (-1)^n \int_0^{+\infty} x e^{-2(n+1)x} \, \mathrm{d}x \\ &= \frac{1}{4} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^2} = \frac{1}{4} \left(\frac{\pi^2}{8} - \frac{\pi^2}{24}\right) = \frac{\pi^2}{48}. \end{aligned} \tag{4.2}$$

The infinite sum is related to the **Basel problem** (see 2024 Final: Question 2).

Tiebreakers Question 1

$$\int \frac{x + 24}{x^3 + 25x^2 + 144x} \, \mathrm{d}x = \frac{1}{6} \ln x - \frac{5}{21} \ln (x + 9) + \frac{1}{14} \ln (x + 16) + C. \tag{5.1}$$

Semifinal #2

Question 1

$$\int \frac{\sqrt{\left(x^6+1\right)\left(x^2+1\right)}}{x^3}\,\mathrm{d}x \tag{6.1}$$

Solution The integrand can be rearranged into

$$\frac{\sqrt{\left(x^6+1\right)\left(x^2+1\right)}}{x^3} = \frac{x^2+1}{x^3}\sqrt{x^4-x^2+1} = \left(1+\frac{1}{x^2}\right)\sqrt{\left(x-\frac{1}{x}\right)^2+1}. \tag{6.2}$$

Therefore, after a **change of variable**, we have

$$t = x - \frac{1}{x}, \qquad I = \int \sqrt{t^2+1}\,\mathrm{d}t = \frac{1}{2}\left(t\sqrt{t^2+1} + \ln\left|t+\sqrt{t^2+1}\right|\right) + C. \tag{6.3}$$

This is equal to

$$I = \frac{1}{2}\left(1-\frac{1}{x^2}\right)\sqrt{x^4-x^2+1} + \frac{1}{2}\operatorname{arcsinh}\left(x-\frac{1}{x}\right) + C. \tag{6.4}$$

Question 2

$$\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\,\mathrm{d}x \tag{7.1}$$

Solution Note that

$$I = \int_0^1 \frac{\ln\left(1-x\right)}{\sqrt{x-x^2}}\,\mathrm{d}x = 2\int_0^{1/2} \frac{\ln\sqrt{x-x^2}}{\sqrt{x-x^2}}\,\mathrm{d}x. \tag{7.2}$$

With the **change of variable** $t = \sqrt{x-x^2}$, we have

$$x = \frac{1-\sqrt{1-4t^2}}{2}, \qquad \mathrm{d}x = \frac{2t\,\mathrm{d}t}{\sqrt{1-4t^2}}, \qquad I = 4\int_0^{1/2}\frac{\ln t}{\sqrt{1-4t^2}}\,\mathrm{d}t. \tag{7.3}$$

With another **change of variable** $2t = \sin u$, we have

$$I = 2\int_0^{\pi/2}\left(\ln\sin u - \ln 2\right)\,\mathrm{d}u = 2\cdot\left(-\frac{\pi}{2}\ln 2 - \frac{\pi}{2}\ln 2\right) = -2\pi\ln 2. \tag{7.4}$$

For the first term, see 2023 Regular Season: Question 5.

Question 3

$$\int_{1}^{+\infty}\left(\sum_{k=0}^{\infty}(-1)^{k}\max\{0,x-k\}\right)^{-2}\,\mathrm{d}x \tag{8.1}$$

Solution It can be shown that

$$\begin{aligned} I &= \int_{1}^{2}\frac{\mathrm{d}x}{[x-(x-1)]^{2}}+\int_{2}^{3}\frac{\mathrm{d}x}{[x-(x-1)+(x-2)]^{2}}+\cdots \\ &= \sum_{n=1}^{+\infty}\left[\int_{2n-1}^{2n}\frac{\mathrm{d}x}{n^{2}}+\int_{2n}^{2n+1}\frac{\mathrm{d}x}{(x-n)^{2}}\right] \\ &= \sum_{n=1}^{+\infty}\frac{1}{n^{2}}+\sum_{n=1}^{+\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{\pi^{2}}{6}+1. \end{aligned} \tag{8.2}$$

Again, the first term is related to the **Basel problem** (see 2024 Final: Question 2).

Question 4

$$\int_{0}^{1}\left\lfloor\log_{2}\left(x-2^{\lfloor\log_{2}x\rfloor}\right)\right\rfloor\,\mathrm{d}x \tag{9.1}$$

Solution Decompose the interval into negative powers of 2, we have

$$\begin{aligned} I &= \sum_{n=1}^{+\infty}\int_{1/2^{n}}^{1/2^{n-1}}\left\lfloor\log_{2}\left(x-2^{-n}\right)\right\rfloor\,\mathrm{d}x \\ &= \sum_{n=1}^{+\infty}\sum_{k=n+1}^{+\infty}\int_{1/2^{n}+1/2^{k}}^{1/2^{n}+1/2^{k-1}}(-k)\,\mathrm{d}x=-\sum_{n=1}^{+\infty}\sum_{k=n+1}^{+\infty}\frac{k}{2^{k}}. \end{aligned} \tag{9.2}$$

Based on the following result

$$S_{n}=\sum_{k=n}^{+\infty}\frac{k}{2^{k}}=\frac{n+1}{2^{n-1}}, \tag{9.3}$$

we eventually obtain

$$I=-\sum_{n=1}^{+\infty}S_{n+1}=-\sum_{n=1}^{+\infty}\frac{n+2}{2^{n}}=-\left(S_{1}+2\right)=-4. \tag{9.4}$$

Note We can also use the **binary representation** of $x\in(0,1)$ (see 2024 Final: Question 5). The integral becomes the opposite of the **expectation** of the index of the second non-zero digit, which is

$$I=-\sum_{k=1}^{+\infty}k\cdot\frac{C_{k-1}^{1}}{2^{k}}=-\sum_{k=1}^{+\infty}\frac{k^{2}-k}{2^{k}}=-4. \tag{9.5}$$