MIT Integration Bee: 2025 Regular Season

Question 1

$$\int_0^{+\infty} \frac{\mathrm{d}x}{(x+1+2\sqrt{x})^2} \tag{1.1}$$

Solution With a **change of variable** $t = \sqrt{x}$, we have

$$I = \int_0^{+\infty} \frac{2t\,\mathrm{d}t}{(t+1)^4} = 2\int_0^{+\infty} \frac{\mathrm{d}t}{(t+1)^3} - 2\int_0^{+\infty} \frac{\mathrm{d}t}{(t+1)^4} = \frac{1}{3}. \tag{1.2}$$

Question 2

$$\int x^2 \cos\left(\csc^{-1} x\right) \mathrm{d}x \tag{2.1}$$

Solution Based on the following trigonometric relations

$$\csc t = x, \qquad \cos t = \frac{\sqrt{x^2-1}}{x}, \tag{2.2}$$

we have

$$I = \int x\sqrt{x^2-1}\,\mathrm{d}x = \frac{1}{3}\left(x^2-1\right)^{3/2} + C. \tag{2.3}$$

Question 3

$$\int_0^{1/2025} \left(\sum_{k=1}^{+\infty} \frac{(2025x)^k e^{-2025x}}{k!}\right) \mathrm{d}x \tag{3.1}$$

Solution Based on the **Taylor series** of e^{2025x}, we have

$$I = \int_0^{1/2025} \left(e^{2025x}-1\right) e^{-2025x}\,\mathrm{d}x = \frac{1}{2025} - \frac{1}{2025}\left(1-\frac{1}{e}\right) = \frac{1}{2025e}. \tag{3.2}$$

Question 4

$$\int \frac{\mathrm{d}x}{1 - x^4} \tag{4.1}$$

Solution

$$I = \frac{1}{2} \int \frac{1}{1 - x^2} \,\mathrm{d}x + \frac{1}{2} \int \frac{1}{1 + x^2} \,\mathrm{d}x = \frac{1}{4} \ln \left| \frac{1 + x}{1 - x} \right| + \frac{1}{2} \arctan x + C. \tag{4.2}$$

Question 5

$$\int_{-\pi/2}^{\pi/2} \cos(20x) \cos(25x) \,\mathrm{d}x \tag{5.1}$$

Solution

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(5x) + \cos(45x)}{2} \,\mathrm{d}x = \frac{1}{5} + \frac{1}{45} = \frac{2}{9}. \tag{5.2}$$

Question 6

$$\int_{-2}^{2} \max\left\{x, x^2, x^3\right\} \mathrm{d}x \tag{6.1}$$

Solution

$$I = \int_{-2}^{0} x^2 \,\mathrm{d}x + \int_{0}^{1} x \,\mathrm{d}x + \int_{1}^{2} x^3 \mathrm{d}x = \frac{8}{3} + \frac{1}{2} + \frac{15}{4} = \frac{83}{12}. \tag{6.2}$$

Question 7

$$\int_{0}^{2025} \left\{\sqrt{x}\right\} \,\mathrm{d}x \tag{7.1}$$

Solution

$$I = \sum_{n=0}^{44} \int_{n^2}^{(n+1)^2} \left(\sqrt{x} - n\right) \mathrm{d}x = \sum_{n=0}^{44} \left(n + \frac{2}{3}\right) = 1020. \tag{7.2}$$

Question 8

$$\int_0^{2\pi} \left|\sin x + \frac{1}{2}\right| \mathrm{d}x \tag{8.1}$$

Solution

$$I = \int_{-\pi/6}^{7\pi/6} \left(\sin x + \frac{1}{2}\right) \mathrm{d}x - \int_{7\pi/6}^{11\pi/6} \left(\sin x + \frac{1}{2}\right) \mathrm{d}x = 2\sqrt{3} + \frac{\pi}{3}. \tag{8.2}$$

Question 9

$$\int x \left(\frac{1}{2} + \ln x\right) \ln(\ln x)\, \mathrm{d}x \tag{9.1}$$

Solution Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \ln x \cdot \ln\ln x\right) = x\,(1 + 2\ln x)\ln\ln x + x. \tag{9.2}$$

Therefore, the original integral is evaluated as

$$I = \frac{1}{2}x^2 \ln x \cdot \ln\ln x - \frac{1}{4}x^2 + C. \tag{9.3}$$

Question 10

$$\int \frac{\cos^4 x - 1}{\sin^8 x}\, \mathrm{d}x \tag{10.1}$$

Solution Using the **reduction formula**

$$I_n = \int \frac{\mathrm{d}x}{\sin^n x}, \qquad (n-1)\, I_n = -\frac{\cot x}{\sin^{n-2} x} + (n-2)\, I_{n-2}, \qquad I_2 = -\cot x + C. \tag{10.2}$$

The original integral becomes

$$\begin{aligned} I &= \int \frac{-\sin^2 x - 2\cos^2 x}{\sin^6 x}\, \mathrm{d}x = I_4 - 2I_6 = -\frac{3}{5}I_4 + \frac{2}{5} \cdot \frac{\cot x}{\sin^4 x} \\ &= \frac{\cot x}{5} \cdot \left(\frac{2}{\sin^4 x} + \frac{1}{\sin^2 x} + 2\right) + C. \end{aligned} \tag{10.3}$$

Question 11

$$\int \sqrt{x+\sqrt{x^2-1}}\,\mathrm{d}x \tag{11.1}$$

Solution Note that

$$x+\sqrt{x^2-1}=\frac{1}{2}\left(\sqrt{x-1}+\sqrt{x+1}\right)^2 \tag{11.2}$$

We thus obtain

$$I=\frac{\sqrt{2}}{3}\left[(x-1)^{3/2}+(x+1)^{3/2}\right]+C. \tag{11.3}$$

Question 12

$$\int \frac{x^3-x}{x^6-1}\,\mathrm{d}x \tag{12.1}$$

Solution Note that

$$\frac{x^3-x}{x^6-1}=\frac{x}{\left(x^2+x+1\right)\left(x^2-x+1\right)}=\frac{x}{\left(x^2+1\right)^2-x^2} \tag{12.2}$$

With the **change of variable** $t=x^2$, we have

$$I=\frac{1}{2}\int \frac{\mathrm{d}t}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{1}{\sqrt{3}}\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)+C. \tag{12.3}$$

Question 13

$$\int \sqrt{x^2-1}\,\mathrm{d}x \tag{13.1}$$

Solution Using **integration by parts**, we have

$$I=x\sqrt{x^2-1}-\int \frac{x^2}{\sqrt{x^2-1}}\,\mathrm{d}x=x\sqrt{x^2-1}-I-\int \frac{\mathrm{d}x}{\sqrt{x^2-1}}. \tag{13.2}$$

Therefore, we have

$$I=\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\operatorname{arccosh}x+C=\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+C. \tag{13.3}$$

Question 14

$$\int \left(\begin{array}{l} \sin x \sin(\sin x) \sin(\cos x) \\ \quad + \cos x \cos(\sin x) \cos(\cos x) \end{array} \right) \mathrm{d}x \tag{14.1}$$

Solution Note that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin(\sin x) = \cos x \cos(\sin x), \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos(\cos x) = \sin x \sin(\cos x). \tag{14.2}$$

We can observe that

$$I = \sin(\sin x) \cdot \cos(\cos x) + C. \tag{14.3}$$

Question 15

$$\int \left(\frac{\ln x}{x} \right)^2 \mathrm{d}x \tag{15.1}$$

Solution Repeatedly using **integration by parts**, we have

$$\begin{aligned} I &= -\frac{\ln^2 x}{x} + \int \frac{2 \ln x}{x^2} \,\mathrm{d}x = -\frac{\ln^2 x}{x} - \frac{2 \ln x}{x} + \int \frac{2}{x^2} \,\mathrm{d}x \\ &= -\frac{1}{x} \left(\ln^2 x + 2 \ln x + 2 \right) + C. \end{aligned} \tag{15.2}$$

Question 16

$$\int \sin^2(2x)\, e^{2x} \,\mathrm{d}x \tag{16.1}$$

Solution

$$I = \frac{1}{4} e^{2x} - \frac{1}{2} \int \cos(4x)\, e^{2x} \,\mathrm{d}x = e^{2x} \left(\frac{1}{4} - \frac{1}{10} \sin 4x - \frac{1}{20} \cos 4x \right) + C. \tag{16.2}$$

The second term is again obtained by repeatedly using **integration by parts**.

Question 17

$$\int_0^{7/2} \sqrt{x + \frac{1}{\sqrt{x + \frac{1}{\sqrt{x+\cdots}}}}}\,\mathrm{d}x \tag{17.1}$$

Solution Denote the integrand as

$$f(x) = \sqrt{x + \frac{1}{\sqrt{x + \frac{1}{\sqrt{x+\cdots}}}}}. \tag{17.2}$$

We can see that $f(x)$ should satisfy

$$f^2 = x + \frac{1}{f}, \qquad x = f^2 - \frac{1}{f}. \tag{17.3}$$

The boundary values are obtained as

$$f(0) = 1, \qquad f\left(\frac{7}{2}\right) = 2. \tag{17.4}$$

With **integration by parts**, we have

$$I = \int_0^{7/2} f\,\mathrm{d}x = xf\Big|_{x=0}^{x=7/2} - \int_1^2 \left(f^2 - \frac{1}{f}\right)\mathrm{d}f = \frac{14}{3} + \ln 2. \tag{17.5}$$

Question 18

$$\int_0^{+\infty} (x+1)^4\, e^{-x^2}\,\mathrm{d}x \tag{18.1}$$

Solution Based on the **reduction formula**

$$I_n = \int_0^{+\infty} x^n\, e^{-x^2}\,\mathrm{d}x = \frac{n-1}{2} I_{n-2}, \qquad \text{with } n \geq 2, \quad I_0 = \frac{\sqrt{\pi}}{2}, \quad I_1 = \frac{1}{2}, \tag{18.2}$$

we have the following results

$$I_0 = \frac{\sqrt{\pi}}{2}, \quad I_1 = \frac{1}{2}, \quad I_2 = \frac{\sqrt{\pi}}{4}, \quad I_3 = \frac{1}{2}, \quad I_4 = \frac{3\sqrt{\pi}}{8}. \tag{18.3}$$

Therefore, the integral is evaluated as

$$I = I_0 + 4I_1 + 6I_2 + 4I_3 + I_4 = 4 + \frac{19}{8}\sqrt{\pi}. \tag{18.4}$$

Question 19

$$\int_0^{\pi/2} \cos(3x)\cos(5x)\cos(7x)\,\mathrm{d}x \tag{19.1}$$

Solution

$$\begin{aligned} I &= \frac{1}{4}\int_0^{\pi/2} (\cos 15x + \cos 9x + \cos 5x + \cos x)\;\mathrm{d}x \\ &= \frac{1}{4}\left(-\frac{1}{15} + \frac{1}{9} + \frac{1}{5} + 1\right) = \frac{14}{45}. \end{aligned} \tag{19.2}$$

Question 20

$$\int_{-1}^{1} e^{2x} \sin(\sinh x)\,\mathrm{d}x \tag{20.1}$$

Solution By **changing the variable** $x \to -x$, we have

$$I = \int_{-1}^{1} e^{2x} \sin(\sinh x)\,\mathrm{d}x = -\int_{-1}^{1} e^{-2x} \sin(\sinh x)\,\mathrm{d}x. \tag{20.2}$$

Therefore, we have

$$I = \int_{-1}^{1} \frac{e^{2x} - e^{-2x}}{2} \sin(\sinh x)\,\mathrm{d}x = \int_{-1}^{1} \sinh(2x)\cdot\sin(\sinh x)\,\mathrm{d}x. \tag{20.3}$$

Using **integration by parts**, we have

$$\int \sinh(2x)\cdot\sin(\sinh x)\,\mathrm{d}x = -2\sinh x\cdot\cos(\sinh x) + 2\sin(\sinh x) + C. \tag{20.4}$$

Therefore, the original integral is evaluated as

$$I = 4\sin(\sinh 1) - 4\sinh 1\cdot\cos(\sinh 1). \tag{20.5}$$