MIT Integration Bee: 2025 Quarterfinal

Quarterfinal #1

Question 1

$$\int_{1}^{+\infty} x^5 e^{-x}\,\mathrm{d}x \tag{1.1}$$

Solution Repeatedly using **integration by parts**, we have

$$I = \frac{1}{e}(1+5+20+60+120+120) = \frac{326}{e}. \tag{1.2}$$

Question 2

$$\int_{0}^{100} \left(\left\lceil\frac{x-1}{3}\right\rceil - \left\lfloor\frac{x+1}{3}\right\rfloor\right)\cdot\left(\left\lceil\frac{x-1}{5}\right\rceil - \left\lfloor\frac{x+1}{5}\right\rfloor\right)\cdot\left(\left\lceil\frac{x-1}{7}\right\rceil - \left\lfloor\frac{x+1}{7}\right\rfloor\right)\mathrm{d}x \tag{2.1}$$

Solution For an integer $n \geq 2$, we have

$$\left\lceil\frac{x-1}{n}\right\rceil - \left\lfloor\frac{x+1}{n}\right\rfloor = \begin{cases} 0, & kn-1 \leq x \leq kn+1, \\ 1, & kn+1 < x < (k+1)n-1, \end{cases} \tag{2.2}$$

Therefore, we only need to find the number of integer m satisfying that all three terms equal to 1 within the interval $[m, m+1)$. The requirements on the integer m are

$$x \equiv 1 \pmod 3, \qquad x \equiv 1,2,3 \pmod 5, \qquad x \equiv 1,2,3,4,5 \pmod 7. \tag{2.3}$$

One trick to find m efficiently is to consider $m+15$, $m+21$ and $m+36$. We can thus find m to be

$$m = 1, 16, 22, 31, 37, 43, 46, 52, 58, 61, 67, 73, 82, 88. \tag{2.4}$$

Therefore, the integral is evaluated to be

$$I = 14. \tag{2.5}$$

Question 3

$$\int \frac{x^2}{\sqrt{4e^{2x} + \left(x^2 + 2x + 2\right)^2}} \, \mathrm{d}x \tag{3.1}$$

Solution

$$I = \int \frac{x^2}{2e^x} \cdot \frac{1}{\sqrt{1 + \left[\frac{1}{2}\left(x^2 + 2x + 2\right) e^{-x}\right]^2}} \, \mathrm{d}x. \tag{3.2}$$

Notice that

$$f(x) = \frac{x^2 + 2x + 2}{2e^x}, \qquad f'(x) = -\frac{x^2}{2e^x}. \tag{3.3}$$

The integral is thus calculated as

$$I = -\int \frac{f'(x)\, \mathrm{d}x}{\sqrt{1 + f^2(x)}} = -\operatorname{arcsinh}\left(\frac{x^2 + 2x + 2}{2e^x}\right) + C. \tag{3.4}$$

Tiebreakers Question 1

$$\int_{-2024}^{2026} x \left[1 + \cos\left(\frac{x-1}{2025}\pi\right)\right] \, \mathrm{d}x \tag{4.1}$$

Solution

$$I = \int_{-2025}^{2025} (x+1) \left[1 + \cos\left(\frac{\pi x}{2025}\right)\right] \, \mathrm{d}x. \tag{4.2}$$

After analyzing the symmetry of each term, we can obtain

$$I = \int_{-2025}^{2025} 1 \, \mathrm{d}x + \int_{-2025}^{2025} \cos\left(\frac{\pi x}{2025}\right) \mathrm{d}x = 4050 + 0 = 4050. \tag{4.3}$$

Tiebreakers Question 2

$$\int_0^2 \lfloor e^x \rfloor \, \mathrm{d}x \tag{5.1}$$

Solution

$$I = \sum_{k=1}^{6} \int_{\ln k}^{\ln(k+1)} k \, \mathrm{d}x + \int_{\ln 7}^{2} 7 \, \mathrm{d}x = 14 - \sum_{k=1}^{7} \ln k = 14 - \ln 5040. \tag{5.2}$$

Tiebreakers Question 3

$$\int_0^{2025} \frac{\lfloor x \rfloor}{\lceil \sqrt{x} \rceil} \, \mathrm{d}x \tag{6.1}$$

Solution

$$\begin{aligned} I &= \sum_{k=1}^{45} \int_{(k-1)^2}^{k^2} \frac{\lfloor x \rfloor}{k} \, \mathrm{d}x = \sum_{k=1}^{45} \frac{1}{k} \cdot \frac{(k-1)^2 + \left(k^2 - 1\right)}{2} \cdot (2k-1) \\ &= \sum_{k=1}^{45} (k-1)(2k-1) = 2 \times \frac{45 \times 46 \times 91}{6} - 3 \times \frac{45 \times 46}{2} + 45 \\ &= 15 \times 23 \times 173 + 45 = 59730. \end{aligned} \tag{6.2}$$

Lightning Question 1

$$\int \frac{\arctan x - x \arctan x}{1 - x + x^2 - x^3} \, \mathrm{d}x \tag{7.1}$$

Solution

$$I = \int \frac{\arctan x}{1 + x^2} \, \mathrm{d}x = \frac{1}{2} (\arctan x)^2 + C. \tag{7.2}$$

Quarterfinal #2

Question 1

$$\lim_{A\to\infty}\int_{-\infty}^{+\infty}\frac{A}{A^2\left(x^3-3x\right)^2+1}\,\mathrm{d}x \tag{8.1}$$

Solution The **Dirac delta function** can be represented by the **Poisson kernel** as

$$\delta(x)=\lim_{\varepsilon\to 0}\eta_\varepsilon(x),\qquad \eta_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}. \tag{8.2}$$

In terms of the integrand, denote $\varepsilon = 1/A$ and we have

$$f_\varepsilon(x)=\frac{\varepsilon}{\left(x^3-3x\right)^2+\varepsilon^2},\qquad \lim_{\varepsilon\to 0} f_\varepsilon(x)=\pi\cdot\delta\left(x^3-3x\right). \tag{8.3}$$

Based on the following property

$$\delta\left[g(x)\right]=\sum_i\frac{\delta(x-x_i)}{|g'(x_i)|},\qquad \text{with } g(x_i)=0, \tag{8.4}$$

the integral is evaluated as

$$I=\pi\int_{-\infty}^{+\infty}\delta\left(x^3-3x\right)\,\mathrm{d}x=\frac{\pi}{3}+\frac{\pi}{6}\times 2=\frac{2\pi}{3}. \tag{8.5}$$

Question 2

$$\int\frac{\mathrm{d}x}{\left[\cos x\cos\left(x+\frac{2\pi}{3}\right)\cos\left(x-\frac{2\pi}{3}\right)\right]^2}\,\mathrm{d}x \tag{9.1}$$

Solution Note that

$$\begin{aligned}\cos x\cos\left(x+\frac{2\pi}{3}\right)\cos\left(x-\frac{2\pi}{3}\right)&=\frac{1}{2}\left(\cos 2x+\cos\frac{4\pi}{3}\right)\cos x\\ =\frac{1}{2}\left(2\cos^2 x-\frac{3}{2}\right)\cos x&=\frac{1}{4}\left(4\cos^3 x-3\cos x\right)=\frac{1}{4}\cos 3x.\end{aligned} \tag{9.2}$$

Therefore, the integral becomes

$$I=16\int\frac{\mathrm{d}x}{\cos^2 3x}=\frac{16}{3}\tan 3x+C. \tag{9.3}$$

Question 3

$$\int_1^{2025} \left(\left\lceil \frac{2025}{\lfloor x \rfloor} \right\rceil - \left\lfloor \frac{2025}{\lceil x \rceil} \right\rfloor \right) \mathrm{d}x \tag{10.1}$$

Solution We can decompose the integral into

$$\begin{aligned} I &= \int_1^2 \left\lceil \frac{2025}{\lfloor x \rfloor} \right\rceil \mathrm{d}x - \int_{2024}^{2025} \left\lfloor \frac{2025}{\lceil x \rceil} \right\rfloor \mathrm{d}x + \sum_{k=2}^{2024} \left(\left\lceil \frac{2025}{k} \right\rceil - \left\lfloor \frac{2025}{k} \right\rfloor \right) \\ &= 2025 - 1 + \sum_{k=2}^{2024} \mathbb{1}\left[k \nmid 2025\right] = 2024 + (2023 - 13) = 4034. \end{aligned} \tag{10.2}$$

The notation $\mathbb{1}\left[k \nmid 2025\right]$ equals to 1 when $k \nmid 2025$ and equals to 0 when k is a factor of 2025. Note that $2025 = 45^2 = 3^4 \times 5^2$, so it has 15 factors in total, and 13 of them lie within $[2, 2024]$.

Tiebreakers Question 1

$$\int (x+1)e^x \ln x \, \mathrm{d}x \tag{11.1}$$

Solution Repeatedly using **integration by parts**, we have

$$\begin{aligned} I &= (x+1)e^x \ln x - \int e^x \left(\ln x + 1 + \frac{1}{x} \right) \mathrm{d}x \\ &= (x+1)e^x \ln x - e^x \ln x + \int \frac{e^x}{x} \mathrm{d}x - e^x - \int \frac{e^x}{x} \mathrm{d}x \\ &= e^x (x \ln x - 1) + C. \end{aligned} \tag{11.2}$$

Quarterfinal #3

Question 1

$$\int_{-\pi/2}^{\pi/2} \sqrt{\sec x - \cos x}\,\mathrm{d}x \tag{12.1}$$

Solution

$$I = \int_{-\pi/2}^{\pi/2} \frac{|\sin x|}{\sqrt{\cos x}}\,\mathrm{d}x = 2\int_0^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,\mathrm{d}x = 4. \tag{12.2}$$

Question 2

$$\int \frac{x}{\sqrt[3]{x^3 - 3x - 2}}\,\mathrm{d}x \tag{13.1}$$

Solution

$$I = \int x\,(x+1)^{-2/3}\,(x-2)^{-1/3}\,\mathrm{d}x. \tag{13.2}$$

Based on the exponent of each term in the denominator, we consider the following function

$$f(x) = (x+1)^{1/3}\,(x-2)^{2/3}, \tag{13.3}$$

and its derivative is

$$\begin{aligned} f(x) &= \frac{1}{3}(x+1)^{-2/3}(x-2)^{2/3} + \frac{2}{3}(x+1)^{1/3}(x-2)^{-1/3} \\ &= \frac{1}{3}\frac{(x-2)+2(x+1)}{\sqrt[3]{(x+1)^2(x-2)}} = \frac{x}{\sqrt[3]{(x+1)^2(x-2)}}. \end{aligned} \tag{13.4}$$

Therefore, we know the result of the integral as

$$I = f(x) + C = (x+1)^{1/3}\,(x-2)^{2/3} + C. \tag{13.5}$$

Question 3

$$\int_0^{2\pi} \left(\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}\right)^2 \mathrm{d}x \tag{14.1}$$

Solution The orthogonality of the **Fourier series** over the interval $[0, 2\pi]$ indicates that all the cross-terms in the integral will not contribute. Therefore, we have

$$I = \sum_{n=0}^{\infty} \int_0^{2\pi} \frac{\cos^2(2^n x)}{4^n}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{1}{4^n} \int_0^{2\pi} \frac{1+\cos\left(2^{n+1}x\right)}{2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{\pi}{4^n} = \frac{4\pi}{3}. \tag{14.2}$$

Quarterfinal #4

Question 1

$$\int_{x=0}^{x=10} x^2 \,\mathrm{d}\left\{x+\frac{1}{2}\right\} \tag{15.1}$$

Solution When $x+1/2$ becomes an integer, the integral at this point is essentially a Dirac delta function. Therefore, we have

$$I=\int_0^{10} x^2\,\mathrm{d}x-\sum_{k=0}^{9}\left(k+\frac{1}{2}\right)^2=\frac{1000}{3}-\frac{9\times 10\times 19}{6}-45-\frac{5}{2}=\frac{5}{6}. \tag{15.2}$$

Question 2

$$\int_0^1 \frac{x^2}{\sqrt{x(1-x)}}\,\mathrm{d}x \tag{16.1}$$

Solution Using the **Beta function**, we have

$$I=B\left(\frac{5}{2},\frac{1}{2}\right)=\frac{3}{8}\Gamma^2\left(\frac{1}{2}\right)=\frac{3\pi}{8}. \tag{16.2}$$

Question 3

$$\int \frac{\mathrm{d}x}{x^8-x^6} \tag{17.1}$$

Solution Note that

$$\frac{1}{x^8-x^6}=\frac{x^6-\left(x^6-1\right)}{x^6\left(x^2-1\right)}=\frac{1}{x^2-1}-\frac{x^4+x^2+1}{x^6}. \tag{17.2}$$

Therefore, the integral is calculated as

$$I=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+C. \tag{17.3}$$