MIT Integration Bee: 2025 Final

Question 1

$$\int \tan x \sqrt{2 + \sqrt{4 + \cos x}}\,\mathrm{d}x \tag{1.1}$$

Solution After several **changes of variable**, the integral becomes doable.

$$\begin{aligned} I &= -\int \frac{1}{u}\sqrt{2 + \sqrt{4 + u}}\,\mathrm{d}u \qquad (u = \cos x) \\ &= -\int \frac{2t}{t^2 - 4}\sqrt{2 + t}\,\mathrm{d}t \qquad \left(t = \sqrt{4 + u}, \quad 2t\,\mathrm{d}t = \mathrm{d}u\right) \\ &= -\int \frac{4z^2\left(z^2 - 2\right)}{\left(z^2 - 2\right)^2 - 4}\,\mathrm{d}z \qquad \left(z = \sqrt{2 + t}, \quad 2z\,\mathrm{d}z = \mathrm{d}t\right) \\ &= -4\int \left(1 + \frac{2}{z^2 - 4}\right)\mathrm{d}z. \end{aligned} \tag{1.2}$$

Therefore, we have

$$I = -4z - 2\ln\left|\frac{z - 2}{z + 2}\right| + C, \qquad \text{with } z = \sqrt{2 + \sqrt{4 + \cos x}}. \tag{1.3}$$

Question 2

$$\int_0^{+\infty} \frac{\mathrm{d}x}{\left(x + 1 + \lfloor 2\sqrt{x} \rfloor\right)^2} \tag{2.1}$$

Solution With a **change of variable** $t = 2\sqrt{x}$, we have

$$\begin{aligned} I &= 4\int_0^{+\infty} \frac{2t\,\mathrm{d}t}{\left(t^2 + 4\lfloor t \rfloor + 4\right)^2} = 4\sum_{k=0}^{+\infty}\int_k^{k+1} \frac{2t\,\mathrm{d}t}{\left(t^2 + 4k + 4\right)^2} \\ &= -4\sum_{k=0}^{+\infty}\left[\frac{1}{t^2 + 4k + 4}\right]_k^{k+1} = 4\sum_{k=0}^{+\infty}\frac{1}{(k + 2)^2} - 4\sum_{k=0}^{+\infty}\frac{1}{(k + 1)(k + 5)}. \end{aligned} \tag{2.2}$$

The first term is related to the **Basel problem**. Eventually, we have

$$I = 4\left(\frac{\pi^2}{6} - 1\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{2\pi^2}{3} - \frac{73}{12}. \tag{2.3}$$

Question 3

$$\int_0^{10} \left\lfloor \left(\frac{1+\sqrt{5}}{2}\right)^{\lfloor x \rfloor} \right\rfloor \mathrm{d}x \tag{3.1}$$

Solution Based on the **Fibonacci sequence**, we have

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad \alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}. \tag{3.2}$$

Therefore, the integral is equivalent to

$$I = \sum_{k=0}^{9} \lfloor \alpha^k \rfloor = \sum_{k=0}^{9} \left\lfloor \sqrt{5}F_k + \beta^k \right\rfloor = \sum_{k=0}^{9} a_k. \tag{3.3}$$

Note that

$$\beta \approx -0.618, \qquad |\beta|^2 < 0.4, \tag{3.4}$$

and the values of the Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3,$$

$$F_5 = 5, \quad F_6 = 8, \quad F_7 = 13, \quad F_8 = 21, \quad F_9 = 34. \tag{3.5}$$

We can neglect the term β^k when k is large, and obtain

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 4, \quad a_4 = 6,$$

$$a_5 = 11, \quad a_6 = 17, \quad a_7 = 29, \quad a_8 = 46, \quad a_9 = 76. \tag{3.6}$$

The integral is thus evaluated as

$$I = \sum_{k=0}^{9} a_k = 193. \tag{3.7}$$

Question 4

$$\int_0^\pi \max\left\{|2\sin x|, |2\cos 2x - 1|\right\}^2 \cdot \min\left\{|\sin 2x|, |\cos 3x|\right\}^2 \mathrm{d}x \tag{4.1}$$

Solution Note that

$$\begin{aligned} \sin 2x &= 2\sin x \cos x, \\ \cos 3x &= 4\cos^3 x - 3\cos x = (2\cos 2x - 1)\cos x. \end{aligned} \tag{4.2}$$

Therefore, denote

$$f(x) = |2\sin x|, \qquad g(x) = |2\cos 2x - 1|, \tag{4.3}$$

the integral can be calculated as

$$\begin{aligned} I &= \int_0^\pi \max\left\{f(x), g(x)\right\}^2 \cdot \min\left\{f(x)|\cos x|, g(x)|\cos x|\right\}^2 \mathrm{d}x \\ &= \int_0^\pi \left[f(x)\, g(x)\, |\cos x|\right]^2 \mathrm{d}x \\ &= \int_0^\pi \left[\sin 2x \cdot (2\cos 2x - 1)\right]^2 \mathrm{d}x \\ &= \int_0^\pi (\sin 4x - \sin 2x)^2 \mathrm{d}x = \pi. \end{aligned} \tag{4.4}$$

Question 5

$$\int_0^1 \left(\sqrt{\frac{1}{4x^2} + \frac{1}{x} - x} - \sqrt{\frac{x^4}{4} - x + 1} - \frac{1}{2x} \right) \mathrm{d}x \tag{5.1}$$

Solution Denote the following function

$$y(x) = \sqrt{\frac{1}{4x^2} + \frac{1}{x} - x} - \frac{1}{2x} = \frac{-1 + \sqrt{1 + 4x - 4x^3}}{2x}. \tag{5.2}$$

We can see that y is the solution of the quadratic equation

$$xy^2 + y + x^2 - 1 = 0, \qquad \text{with } y(0) = 1, \quad y(1) = 0. \tag{5.3}$$

The **inverse** of $y(x)$ can be obtained as

$$x(y) = \frac{-y^2 + \sqrt{y^4 - 4y + 4}}{2}, \tag{5.4}$$

where the + sign in the numerator is determined by the boundary values. Based on **integration by parts**, we have

$$\begin{aligned} \int_0^1 y(x)\,\mathrm{d}x &= xy\Big|_{x=0}^{x=1} - \int_1^0 x(y)\,\mathrm{d}y \\ &= \int_0^1 \left(-\frac{y^2}{2} + \sqrt{\frac{y^4}{4} - y + 1} \right) \mathrm{d}y. \end{aligned} \tag{5.5}$$

Now, the original integral simply becomes

$$I = -\frac{1}{2} \int_0^1 y^2\,\mathrm{d}y = -\frac{1}{6}. \tag{5.6}$$