MIT Integration Bee: 2024 Semifinal

Semifinal #1

Question 1

$$\int_{-\infty}^{+\infty} \frac{\left(x^3-4x\right)\sin x + \left(3x^2-4\right)\cos x}{\left(x^3-4x\right)^2+\cos^2 x}\,\mathrm{d}x \tag{1.1}$$

Solution Denote the following functions

$$f(x) = \cos x, \qquad g(x) = x^3 - 4x, \qquad t(x) = \frac{f(x)}{g(x)}. \tag{1.2}$$

We thus have

$$I = -\int_{-\infty}^{+\infty} \frac{f'g - fg'}{f^2+g^2}\,\mathrm{d}x = -\int_{-\infty}^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t. \tag{1.3}$$

Note that there are **singularities** $x = 0, \pm 2$ after the substitution. Because $\cos 2 < 0$, we have

$$\begin{aligned}
&t(-\infty) \to 0, \quad t(-2^-) \to +\infty, \quad t(-2^+) \to -\infty, \quad t(0^-) \to +\infty, \\
&t(0^+) \to -\infty, \quad t(2^-) \to +\infty, \quad t(2^+) \to -\infty, \quad t(\infty) \to 0.
\end{aligned}$$

Therefore, the proper way to evaluate the integral should be

$$\begin{aligned}
I &= -\left(\arctan t\Big|_{x=-\infty}^{x=-2^-} + \arctan t\Big|_{x=-2^+}^{x=0^-} + \arctan t\Big|_{x=0^+}^{x=2^-} + \arctan t\Big|_{x=2^+}^{x=+\infty}\right) \\
&= -\left[\arctan(+\infty) - 0 + 2\arctan(+\infty) - 2\arctan(-\infty) + 0 - \arctan(-\infty)\right] \\
&= -6\arctan(+\infty) = -3\pi.
\end{aligned} \tag{1.4}$$

Question 2

$$\int_0^{\infty} \frac{xe^{-2x}}{e^{-x}+1}\,\mathrm{d}x \tag{2.1}$$

Solution With a **change of variable**, we have

$$\begin{aligned} I &= \int_0^{\infty} \frac{xe^{-2x}}{e^{-x}+1}\,\mathrm{d}x \\ &= -\int_0^1 \frac{t\ln t}{t+1}\,\mathrm{d}t \qquad \left(t = e^{-x}, \quad x = -\ln t, \quad \mathrm{d}x = -\frac{\mathrm{d}t}{t}\right). \end{aligned} \tag{2.2}$$

Using **Taylor series**, we have

$$I = -\int_0^1 \ln t\,\mathrm{d}t + \int_0^1 \frac{\ln t}{t+1}\,\mathrm{d}t = 1 + \sum_{n=0}^{+\infty}(-1)^n \int_0^1 t^n \ln t\,\mathrm{d}t. \tag{2.3}$$

Based on the following results (see 2024 Final: Question 2)

$$\int_0^1 t^n \ln t\,\mathrm{d}t = -\frac{1}{(n+1)^2} \qquad n \in \mathbb{N}^*, \tag{2.4}$$

and the **Basel problem**, we have

$$I = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = 1 - \frac{\pi^2}{12}. \tag{2.5}$$

Question 3

$$\int_0^{\pi/2} \sin\left(\cot^2 x\right) \sec^2 x \,\mathrm{d}x \tag{3.1}$$

Solution With a **change of variable**, we have

$$\begin{aligned} I &= \int_0^{\pi/2} \sin\left(\cot^2 x\right) \sec^2 x \,\mathrm{d}x \\ &= \int_0^{+\infty} \frac{\sin t^2}{t^2} \,\mathrm{d}t \qquad \left(t = \cot x, \quad \tan x = \frac{1}{t}, \quad \sec^2 x \,\mathrm{d}x = -\frac{\mathrm{d}t}{t^2}\right). \end{aligned} \tag{3.2}$$

Integration by parts leads to the **Fresnel integral**

$$\begin{aligned} I &= \int_0^{+\infty} \frac{\sin t^2}{t^2} \,\mathrm{d}t = -\frac{\sin t^2}{t}\bigg|_0^{+\infty} + 2\int_0^{+\infty} \cos t^2 \,\mathrm{d}t \\ &= 2\int_0^{+\infty} \cos t^2 \,\mathrm{d}t = \sqrt{\frac{\pi}{2}}. \end{aligned} \tag{3.3}$$

Note One particular result for the definite Fresnel integral is

$$\int_0^{+\infty} \cos\left(x^\alpha\right) \mathrm{d}x = \Gamma\left(1 + \frac{1}{a}\right) \cos\left(\frac{\pi}{2\alpha}\right), \tag{3.4}$$

$$\int_0^{+\infty} \sin\left(x^\alpha\right) \mathrm{d}x = \Gamma\left(1 + \frac{1}{a}\right) \sin\left(\frac{\pi}{2\alpha}\right), \qquad \text{for } \alpha > 1. \tag{3.5}$$

These are obtained by evaluating the integral of e^{-z^α} using the contour in Fig. 1.

$$e^{i\theta} \int_0^{+\infty} e^{-ir^\alpha} \,\mathrm{d}r = \int_0^{+\infty} e^{-x^\alpha} \,\mathrm{d}x = \frac{1}{\alpha} \int_0^{+\infty} t^{\frac{1}{\alpha}-1} e^{-t} \,\mathrm{d}t = \Gamma\left(1 + \frac{1}{\alpha}\right). \tag{3.6}$$

The real and imaginary parts correspond to Eqs (3.4) and (3.5), respectively.

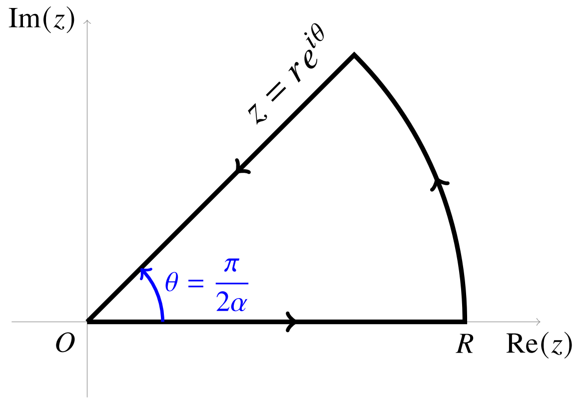

Fig. 1 Sector-shaped contour in the upper plane to evaluate the Fresnel integral.

Question 4

$$\int \cosh^2(3x)\tanh(2x)\,\mathrm{d}x \tag{4.1}$$

Solution Using the following identities for the hyperbolic functions

$$\cosh^2 x = \frac{1+\cosh(2x)}{2}, \qquad \cosh(3x) = 4\cosh^3 x - 3\cosh x \tag{4.2}$$

we have

$$\begin{aligned}
I &= \frac{1}{2}\int \tanh(2x)\,\mathrm{d}x + \frac{1}{2}\int \cosh(6x)\tanh(2x)\,\mathrm{d}x \\
&= \frac{1}{4}\ln|\cosh(2x)| + \int \left[2\cosh^2(2x) - \frac{3}{2}\right]\sinh(2x)\,\mathrm{d}x \\
&= \frac{1}{4}\ln|\cosh(2x)| + \frac{1}{3}\cosh^3(2x) - \frac{3}{4}\cosh(2x) + C \\
&= \frac{1}{4}\ln|\cosh(2x)| + \frac{1}{12}\cosh(6x) - \frac{1}{2}\cosh(2x) + C.
\end{aligned} \tag{4.3}$$

Tiebreakers Question 1

$$\int \sec^5 x \, dx \tag{5.1}$$

Solution Note that

$$(\tan x)' = \sec^2 x, \qquad (\sec x)' = \tan x \sec x, \qquad \tan^2 x = \sec^2 x - 1. \tag{5.2}$$

We can first obtain the **reduction formula** for the general integral. Since we have

$$\begin{aligned} I_n = \int \sec^n x \, dx &= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \left(I_n - I_{n-2} \right), \end{aligned} \tag{5.3}$$

the **recurrence relation** is

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}, \qquad n \geq 3. \tag{5.4}$$

When $n = 1, 2$, we have

$$I_1 = \int \sec x \, dx = \ln |\sec x + \tan x| + C, \qquad I_2 = \int \sec^2 x \, dx = \tan x + C. \tag{5.5}$$

Therefore, we can obtain

$$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C. \tag{5.6}$$

Tiebreakers Question 2

$$\int_{-\infty}^{+\infty} \operatorname{sech}\left(2x + 1 - \frac{1}{x-1} - \frac{2}{x+1}\right) \mathrm{d}x \tag{6.1}$$

Solution The **Glasser's master theorem** states that the identity

$$\mathrm{PV} \int_{-\infty}^{+\infty} F(\phi(x)) \, \mathrm{d}x = \mathrm{PV} \int_{-\infty}^{+\infty} F(x) \, \mathrm{d}x \tag{6.2}$$

holds for any integrable function $F(x)$ and $\phi(x)$ of the form

$$\phi(x) = x - a - \sum_{n=1}^{N} \frac{|\alpha_n|}{x - \beta_n}, \tag{6.3}$$

with a, α_n and β_n being arbitrary real constants. Consequently, the integral is equivalent to

$$I = \int_{-\infty}^{+\infty} \operatorname{sech}(2x) \, \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{2e^{2x}}{e^{4x} + 1} \, \mathrm{d}x = \left(\arctan e^{2x}\right) \Big|_{-\infty}^{+\infty} = \frac{\pi}{2}. \tag{6.4}$$

Semifinal #2

Question 1

$$\int_0^{+\infty} \frac{\sin(x)\sin(2x)\sin(3x)}{x^3}\,\mathrm{d}x \tag{7.1}$$

Solution Using the trigonometric identities, we have

$$I = \frac{1}{2}\int_0^{+\infty} \frac{(\cos x - \cos 3x)\sin 3x}{x^3}\,\mathrm{d}x = \frac{1}{4}\int_0^{+\infty} \frac{\sin 2x + \sin 4x - \sin 6x}{x^3}\,\mathrm{d}x. \tag{7.2}$$

We define the following integral

$$F(\alpha) = \int_0^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3}\,\mathrm{d}x. \tag{7.3}$$

Using **integration by parts**, we have

$$\begin{aligned} F(\alpha) &= -\frac{\alpha - \sin \alpha x}{2x^2}\bigg|_0^{+\infty} + \frac{\alpha}{2}\int_0^{+\infty} \frac{1-\cos \alpha x}{x^2}\,\mathrm{d}x \\ &= -\frac{\alpha}{2}\frac{1-\cos \alpha x}{x}\bigg|_0^{+\infty} + \frac{\alpha^2}{2}\int_0^{+\infty} \frac{\sin \alpha x}{x}\,\mathrm{d}x = \frac{\pi\alpha^2}{4}. \end{aligned} \tag{7.4}$$

Finally, we obtain

$$I = \frac{1}{4}\left[-F(2) - F(4) + F(6)\right] = \pi. \tag{7.5}$$

Question 2

$$\int (1 + \ln x)(1 + \ln\ln x)\,\mathrm{d}x \tag{8.1}$$

Solution Note that

$$(x\ln x)' = 1 + \ln x, \qquad (1 + \ln\ln x)' = \frac{1}{x\ln x}. \tag{8.2}$$

We have

$$\begin{aligned} I &= \int (1 + \ln\ln x)\,\mathrm{d}(x\ln x) \\ &= x\ln x\,(1 + \ln\ln x) - \int 1\,\mathrm{d}x \\ &= -x + x\ln x + (x\ln x)\ln\ln x + C. \end{aligned} \tag{8.3}$$

Question 3

$$\int_0^{+\infty} \frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2} \, \mathrm{d}x \tag{9.1}$$

Solution It can be shown that

$$f(x) = \left(x + \frac{1}{2x}\right)^{-1} = \frac{x\left(x^2+\frac{1}{2}\right)}{\left(x^2+\frac{1}{2}\right)^2}, \qquad f'(x) = \frac{\frac{1}{2}-x^2}{\left(x^2+\frac{1}{2}\right)^2}. \tag{9.2}$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2} f(x)\right] = e^{-x^2}\left[f'(x) - 2x f(x)\right] = e^{-x^2} \cdot \frac{1-2\left(x^2+\frac{1}{2}\right)^2}{\left(x^2+\frac{1}{2}\right)^2}, \tag{9.3}$$

which implies that

$$\frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2} f(x)\right] + 2e^{-x^2}. \tag{9.4}$$

The integral can thus be evaluated as

$$I = \left[e^{-x^2} f(x)\right]_0^{\infty} + 2\int_0^{+\infty} e^{-x^2} \, \mathrm{d}x = 2\int_0^{+\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}. \tag{9.5}$$

Question 4

$$\int \tan x \, \sec^2 x \, \cos(2x) \, e^{2\cos x} \, \mathrm{d}x \tag{10.1}$$

Solution With a **change of variable** $t = \cos x$, we have

$$I = -\int \frac{2t^2 - 1}{t^3} e^{2t} \, \mathrm{d}t \tag{10.2}$$

We can also obtain the **reduction formula** for the following integral

$$I_n = \int \frac{e^{2t}}{t^n} \mathrm{d}t = -\frac{1}{n-1}\frac{e^{2t}}{t^{n-1}} + \frac{2}{n-1} I_{n-1}. \tag{10.3}$$

Finally, we have

$$\begin{aligned} I &= I_3 - 2I_1 = -\frac{1}{2}\frac{e^{2t}}{t^2} + I_2 - 2I_1 \\ &= -\frac{1}{2}\frac{e^{2t}}{t^2} - \frac{e^{2t}}{t} + 2I_1 - 2I_1 = -\left(\frac{1}{2t^2} + \frac{1}{t}\right) e^{2t} + C \\ &= -\left(\frac{\sec^2 x}{2} + \sec x\right) e^{2\cos x} + C. \end{aligned} \tag{10.4}$$