MIT Integration Bee: 2024 Regular Season

Question 1

$$\int_1^{2024} \lfloor \log_{43}(x) \rfloor \, dx \tag{1.1}$$

Solution

$$I = \int_1^{43} 0 \, dx + \int_{43}^{1849} 1 \, dx + \int_{1849}^{2024} 2 \, dx = 1806 + 350 = 2156. \tag{1.2}$$

Question 2

$$\int \frac{dx}{x^{2024} - x^{4047}} \tag{2.1}$$

Solution

$$\begin{aligned}
I &= \int \left(\frac{1}{x^{2024}} + \frac{1}{x} \cdot \frac{1}{1 - x^{2023}} \right) dx \\
&= \int \left(\frac{1}{x^{2024}} + \frac{1}{x} + \frac{x^{2022}}{1 - x^{2023}} \right) dx \\
&= -\frac{1}{2023 \, x^{2023}} + \ln x - \frac{1}{2023} \ln \left(1 - x^{2023} \right) + C.
\end{aligned} \tag{2.2}$$

Question 3

$$\int_0^1 x^2 (1 - x)^{2024} \, dx \tag{3.1}$$

Solution

$$I = B(3, 2025) = \frac{2 \cdot 2024!}{2027!} = \frac{2}{2025 \times 2026 \times 2027}. \tag{3.2}$$

Question 4

$$\int \frac{2023x + 1}{x^2 + 2024}\, dx \tag{4.1}$$

Solution

$$
\begin{aligned}
I &= 2023 \int \frac{x}{x^2 + 2024}\, dx + \int \frac{1}{x^2 + 2024}\, dx \\
&= \frac{2023}{2} \ln\left(x^2 + 2024\right) + \frac{1}{\sqrt{2024}} \arctan \frac{x}{\sqrt{2024}} + C
\end{aligned}
\tag{4.2}
$$

Question 5

$$\int_0^{\pi/2} \sec^2(x)\, e^{-\sec^2(x)}\, dx \tag{5.1}$$

Solution Notice that $\sec^2 x = \tan^2 x + 1$. With a change of variable $t = \tan x$, we have

$$I = \int_0^\infty e^{-t^2 - 1}\, dt = \frac{\sqrt{\pi}}{2e}. \tag{5.2}$$

Question 6

$$\int \cot x\, \cot 2x\, dx \tag{6.1}$$

Solution

$$
\begin{aligned}
I &= \int \frac{\cos 2x}{2 \sin^2 x}\, dx = \int \frac{1 - 2 \sin^2 x}{2 \sin^2 x}\, dx \\
&= -\frac{1}{2} \cot x - x + C.
\end{aligned}
\tag{6.2}
$$

Question 7

$$\int \frac{\sinh^2 x}{\tanh 2x}\, dx \tag{7.1}$$

Solution

$$
\begin{aligned}
I &= \int \frac{\sinh x \cosh 2x}{2 \cosh x}\, dx = \frac{1}{2} \int \sinh 2x\, dx - \frac{1}{2} \int \tanh x\, dx \\
&= \frac{1}{4} \cosh 2x - \frac{1}{2} \ln(\cosh x) + C.
\end{aligned}
\tag{7.2}
$$

Question 8

$$\int \arctan\left(\sqrt{x}\right) dx \tag{8.1}$$

Solution

$$\begin{aligned} I &= x \arctan\left(\sqrt{x}\right) - \frac{1}{2} \int \frac{\sqrt{x}}{1+x} dx \\ &= x \arctan\left(\sqrt{x}\right) - \int \frac{t^2}{1+t^2} dt \qquad \left(t = \sqrt{x}, \quad x = t^2, \quad dx = 2t\, dt\right) \\ &= (x+1) \arctan\left(\sqrt{x}\right) - \sqrt{x} + C. \end{aligned} \tag{8.2}$$

Question 9

$$\int_0^\infty \frac{x \ln x}{x^4 + 1} dx \tag{9.1}$$

Solution With the change of variable $t = 1/x$, we find that

$$I = \int_0^\infty \frac{x \ln x}{x^4 + 1} dx = -\int_0^\infty \frac{t \ln t}{t^4 + 1} dt = -I. \tag{9.2}$$

Therefore, the result is $I = 0$.

Question 10

$$\int_0^{10} \lfloor x \lfloor x \rfloor \rfloor\, dx \tag{10.1}$$

Solution We first study the integral over a general interval $[n, n+1]$ for any $n \in \mathbb{N}$, which is

$$I_n = \int_n^{n+1} \lfloor x \lfloor x \rfloor \rfloor\, dx = \int_n^{n+1} \lfloor nx \rfloor\, dx. \tag{10.2}$$

It is trivial that $I_0 = 0$. For $n \geq 1$, the integrand is a piecewise constant function, with its value ranging from n^2 to $n^2 + n - 1$. The integral is equivalent to the average of all these values, giving

$$I_n = \frac{1}{n}\left[n^2 + (n^2 + 1) + \cdots + (n^2 + n - 1)\right] = n^2 + \frac{n-1}{2}, \qquad \text{for } n \geq 1. \tag{10.3}$$

Therefore, we obtain

$$\begin{aligned} I &= \sum_{k=0}^9 I_k = \left(1^2 + 2^2 + \cdots + 9^2\right) + \frac{1 + 2 + \cdots + 8}{2} \\ &= \frac{9 \times 10 \times 19}{6} + 18 = 303. \end{aligned} \tag{10.4}$$

Question 11

$$\int_0^1 e^{-x} \sqrt{1 + \cot^2\left(\arccos e^{-x}\right)} \, dx \qquad (11.1)$$

Solution   With the change of variable $t = \arccos e^{-x}$, we have

$$t(0) = 0, \qquad t(1) = \arccos e^{-1}, \qquad x = -\ln|\cos t|, \qquad dx = \tan t \, dt. \qquad (11.2)$$

Therefore, the integral becomes

$$
\begin{aligned}
I &= \int_0^{\arccos e^{-1}} \cos t \sqrt{1 + \cot^2 t} \cdot \tan t \, dt \\
&= \int_0^{\arccos e^{-1}} \sqrt{\sin^2 t + \cos^2 t} \, dt = \arccos e^{-1}.
\end{aligned}
\qquad (11.3)
$$

Question 12

$$\int_1^3 \frac{1 + \frac{1+\cdots}{x+\cdots}}{x + \frac{1+\cdots}{x+\cdots}} \, dx \qquad (12.1)$$

Solution   Denote the integrand as $f(x)$. Therefore, we have

$$f(x) = \frac{1 + f(x)}{x + f(x)}, \qquad f(x) = \frac{1 - x + \sqrt{(x-1)^2 + 4}}{2}. \qquad (12.2)$$

The integral can be evaluated as

$$
\begin{aligned}
I &= -1 + \frac{1}{2} \int_0^2 \sqrt{x^2 + 4} \, dx \\
&= -1 + \frac{1}{2} \left(\frac{1}{2} x \sqrt{x^2 + 4} + 2 \ln\left| \frac{x + \sqrt{x^2 + 4}}{2} \right| \right) \Bigg|_0^2 \\
&= -1 + \sqrt{2} + \ln\left(1 + \sqrt{2}\right).
\end{aligned}
\qquad (12.3)
$$

Note   A better way is to use **integration by parts**

$$x = \frac{1 + f - f^2}{f}, \qquad f(1) = 1, \qquad f(3) = \sqrt{2} - 1. \qquad (12.4)$$

The integral can be alternatively calculated as

$$I = x f \big|_1^3 - \int_1^{\sqrt{2}-1} \left(\frac{1}{f} + 1 - f \right) df = \sqrt{2} - 1 - \ln\left(\sqrt{2} - 1\right). \qquad (12.5)$$

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Question 13

$$\int_0^1 \frac{2x(1-x)^2}{1+x^2} \, dx \tag{13.1}$$

Solution

$$I = \int_0^1 \left(2x - 4 + \frac{4}{1+x^2} \right) dx = \pi - 3. \tag{13.2}$$

Question 14

$$\int e^{e^x + 3x} \, dx \tag{14.1}$$

Solution With the change of variable $t = e^x$, we have

$$\begin{aligned}
I = \int e^{e^x + 3x} \, dx &= \int e^{t + 3\ln t} \, \frac{dt}{t} \\
&= \int t^2 e^t \, dt = (t^2 - 2t + 2)e^t + C \\
&= \left(e^{2x} - 2e^x + 2 \right) e^{e^x} + C.
\end{aligned} \tag{14.2}$$

Question 15

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} 2 \left(1 - \frac{|x|}{\sqrt{3}} \right) dx \tag{15.1}$$

Solution

$$I = 2\sqrt{3} - \frac{4}{\sqrt{3}} \int_0^{\sqrt{3}/2} x \, dx = \frac{3\sqrt{3}}{2}. \tag{15.2}$$

Question 16

$$\int \frac{\ln\left(1+x^2\right)}{x^2} \, dx \tag{16.1}$$

Solution

$$\begin{aligned}
I &= -\frac{\ln\left(1+x^2\right)}{x} + 2 \int \frac{dx}{1+x^2} \\
&= -\frac{\ln\left(1+x^2\right)}{x} + 2\arctan x + C.
\end{aligned} \tag{16.2}$$

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Question 17

$$\int 2^x x^2 \, dx \tag{17.1}$$

Solution

$$
\begin{aligned}
I = \int 2^x x^2 \, dx &= \frac{1}{\ln 2} \left(2^x x^2 - 2 \int 2^x x \, dx \right) \\
&= \frac{2^x x^2}{\ln 2} - \frac{2}{\ln^2 2} \left(2^x x - \frac{2^x}{\ln 2} \right) + C \\
&= \frac{2^x}{\ln^3 2} \left(x^2 \ln^2 2 - 2x \ln 2 + 2 \right) + C.
\end{aligned} \tag{17.2}
$$

Question 18

$$\int_0^1 \sqrt{x^8 - x^6 + x^4} \cdot \sqrt{1 + x^2} \, dx \tag{18.1}$$

Solution With the change of variable $t = x^3$, we have

$$
\begin{aligned}
I = \int_0^1 x^2 \sqrt{1 + x^6} \, dx &= \frac{1}{3} \int_0^1 \sqrt{1 + t^2} \, dt \\
&= \frac{1}{6} \left[t\sqrt{1 + t^2} + \ln\left(t + \sqrt{1 + t^2} \right) \right]_0^1 = \frac{1}{6} \left[\sqrt{2} + \ln\left(1 + \sqrt{2} \right) \right].
\end{aligned} \tag{18.2}
$$

Question 19

$$\int_1^\infty \frac{e^x + xe^x}{x^2 e^{2x} - 1} \, dx \tag{19.1}$$

Solution With the change of variable $t = xe^x$, we have

$$
\begin{aligned}
I = \int_1^\infty \frac{d\left(xe^x \right)}{x^2 e^{2x} - 1} &= \int_e^\infty \frac{dt}{t^2 - 1} \\
&= \frac{1}{2} \ln\left(\frac{t - 1}{t + 1} \right) \Big|_e^\infty = \frac{1}{2} \ln\left(\frac{e + 1}{e - 1} \right).
\end{aligned} \tag{19.2}
$$

Question 20

$$\int_0^\infty \left(80x^3 - 60x^4 + 14x^5 - x^6\right) e^{-x} \, \mathrm{d}x \tag{20.1}$$

Solution For each term, we can obtain

$$\begin{aligned} I_n &= \int_0^\infty a_n x^n e^{-x} \, \mathrm{d}x = \int_0^\infty n a_n x^{n-1} e^{-x} \, \mathrm{d}x = \cdots \\ &= \int_0^\infty n! \cdot a_n e^{-x} \, \mathrm{d}x = n! \cdot a_n. \end{aligned} \tag{20.2}$$

Therefore, we have

$$I = 80 \times 6 - 60 \times 24 + 14 \times 120 - 720 = -80 \times 12 + 120 \times 8 = 0. \tag{20.3}$$