MIT Integration Bee: 2024 Final

Question 1

$$\int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\,\mathrm{d}x \tag{1.1}$$

Solution Denote the following integrals

$$I = \int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\,\mathrm{d}x, \qquad J = \int \frac{e^{x/2}\sin x}{\sqrt[3]{3\cos x + 4\sin x}}\,\mathrm{d}x. \tag{1.2}$$

Now consider the following function $f(x)$ and its derivative

$$f(x) = (3\cos x + 4\sin x)^{2/3} = \frac{3\cos x + 4\sin x}{\sqrt[3]{3\cos x + 4\sin x}}, \qquad f'(x) = \frac{2}{3}\,\frac{4\cos x - 3\sin x}{\sqrt[3]{3\cos x + 4\sin x}}. \tag{1.3}$$

We can obtain

$$\int f(x)\,\mathrm{d}e^{x/2} = \int \frac{1}{2}e^{x/2} f(x)\,\mathrm{d}x = \frac{3}{2}I + 2J, \tag{1.4}$$

$$\int e^{x/2}\,\mathrm{d}f(x) = \int e^{x/2} f'(x)\,\mathrm{d}x = \frac{8}{3}I - 2J. \tag{1.5}$$

Therefore, based on the **product rule** we have

$$\begin{aligned} I &= \frac{6}{25}\left[\int f(x)\,\mathrm{d}e^{x/2} + \int e^{x/2}\,\mathrm{d}f(x)\right] \\ &= \frac{6}{25}\,e^{x/2}\,(3\cos x + 4\sin x)^{2/3} + C. \end{aligned} \tag{1.6}$$

Question 2

$$\int_0^\infty \frac{\ln(2e^x - 1)}{e^x - 1}\,\mathrm{d}x \tag{2.1}$$

Solution With several **changes of variables**, we have

$$\begin{aligned} I &= \int_0^\infty \frac{\ln(2e^x - 1)}{e^x - 1}\,\mathrm{d}x \\ &= \int_1^{+\infty} \frac{2\ln t}{t^2 - 1}\,\mathrm{d}t \qquad \left(t = 2e^x - 1, \quad x = \ln\left(\frac{t+1}{2}\right), \quad \mathrm{d}x = \frac{\mathrm{d}t}{t+1}\right) \\ &= \int_0^1 \frac{2\ln t}{t^2 - 1}\,\mathrm{d}t \qquad \left(t \to \frac{1}{t}, \quad \mathrm{d}t \to -\frac{\mathrm{d}t}{t^2}\right). \end{aligned} \tag{2.2}$$

Using **Taylor series**, we have

$$I = -2\int_0^1 \ln t \sum_{n=0}^{+\infty} t^{2n}\,\mathrm{d}t = -2\sum_{n=0}^{+\infty}\int_0^1 t^{2n}\ln t\,\mathrm{d}t. \tag{2.3}$$

As we have the following result

$$\int_0^1 t^n \ln t\,\mathrm{d}t = \frac{t^{n+1}}{n+1}\ln t\,\Big|_0^1 - \int_0^1 \frac{t^n}{n+1}\,\mathrm{d}t = -\frac{1}{(n+1)^2} \qquad n \in \mathbb{N}^*, \tag{2.4}$$

we finally obtain

$$I = 2\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = 2\cdot\frac{\pi^2}{8} = \frac{\pi^2}{4}. \tag{2.5}$$

Note The **Basel problem** directly gives the result of the infinite sum in Eq. (2.5) as follows

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24}, \qquad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2} - \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8}. \tag{2.6}$$

Question 3

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^4 + x^3 + x^2 + x + 1} \tag{3.1}$$

Solution Directly using the **residue theorem** on the contour in Fig. 1, we have

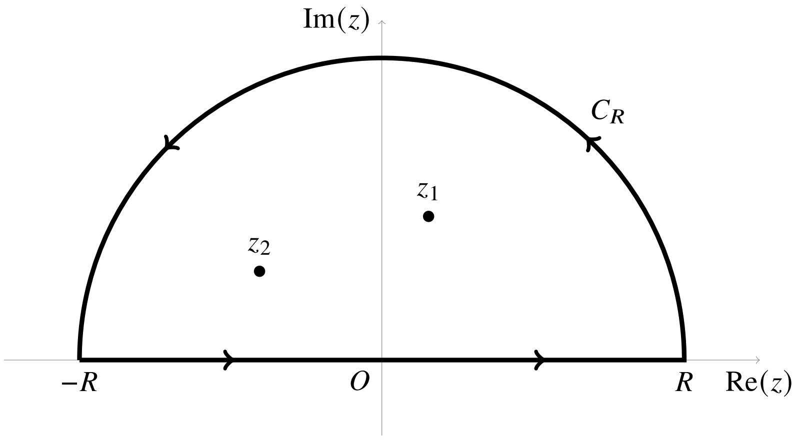

Fig. 1 Semi-circle contour in the upper plane to evaluate the integral over the real line. z_1 and z_2 are simple poles.

$$I = \int_{-\infty}^{+\infty} \frac{x-1}{x^5-1} \mathrm{d}x = 2\pi i \left[\mathrm{Res}\,(f, z_1) + \mathrm{Res}\,(f, z_2)\right] \tag{3.2}$$

where

$$f(z) = \frac{z-1}{z^5-1}, \qquad z_1 = \exp\left(\frac{2\pi i}{5}\right), \qquad z_2 = \exp\left(\frac{4\pi i}{5}\right) = z_1^2. \tag{3.3}$$

The residues are calculated as

$$\mathrm{Res}\,(f, z_1) = \lim_{z \to z_1} \frac{(z-z_1)(z-1)}{z^5-1} = \frac{z_1-1}{5z_1^4} = \frac{z_1-1}{5\bar{z}_1} = \frac{z_1^2-z_1}{5} = \frac{z_2-z_1}{5}, \tag{3.4}$$

$$\mathrm{Res}\,(f, z_2) = \lim_{z \to z_2} \frac{(z-z_2)(z-1)}{z^5-1} = \frac{z_2-1}{5z_2^4} = \frac{z_2-1}{5\bar{z}_2} = \frac{z_2^2-z_2}{5} = \frac{\bar{z}_1-z_2}{5}. \tag{3.5}$$

Finally, we obtain

$$I = \frac{2\pi i}{5}(\bar{z}_1 - z_1) = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right) = \frac{4\pi}{5}\frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{\sqrt{10+2\sqrt{5}}}{5}\pi. \tag{3.6}$$

Question 4

$$\int_{-1/3}^{1}\left(\sqrt[3]{1+\sqrt{1-x^3}}+\sqrt[3]{1-\sqrt{1-x^3}}\right)\,\mathrm{d}x \tag{4.1}$$

Solution Denote

$$a(x)=\sqrt[3]{1+\sqrt{1-x^3}},\qquad b(x)=\sqrt[3]{1-\sqrt{1-x^3}},\qquad y(x)=a(x)+b(x). \tag{4.2}$$

We notice that

$$a^3+b^3=2,\quad ab=x\qquad\Longrightarrow\qquad y^3=a^3+b^3+3ab(a+b)=2+3xy. \tag{4.3}$$

Therefore, using **integration by parts**, we have

$$I=\int_{-1/3}^{1}y(x)\,\mathrm{d}x=xy\,\Big|_{x=-1/3}^{x=1}-\int_{y(-1/3)}^{y(1)}x(y)\,\mathrm{d}y. \tag{4.4}$$

Based on the following results

$$y^3-3xy-2=0\qquad\Longrightarrow\qquad y(1)=2,\quad y\left(-\frac{1}{3}\right)=1,\quad x(y)=\frac{y^3-2}{3y}, \tag{4.5}$$

we finally obtain

$$I=\frac{7}{3}-\int_{1}^{2}\frac{y^3-2}{3y}\,\mathrm{d}y=\frac{14}{9}+\frac{2}{3}\ln 2. \tag{4.6}$$

Question 5

$$\int_0^1 \max_{n\in\mathbb{N}} \left[\frac{1}{2^n}\left(\lfloor 2^n x\rfloor - \left\lfloor 2^n x - \frac{1}{4}\right\rfloor\right)\right] \mathrm{d}x \tag{5.1}$$

Solution For a real number $x \in (0,1)$, its **binary representation** is

$$x = \sum_{k=1}^{+\infty} \frac{a_k}{2^k} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_k}{2^k} + \cdots, \qquad \text{with } a_k \in \{0,1\}. \tag{5.2}$$

We need to evaluate the difference between the integer parts of the following two expressions

$$2^n x = 2^{n-1}a_1 + \cdots + a_n + \frac{a_{n+1}}{2} + \frac{a_{n+2}}{2^2} + \frac{a_{n+3}}{2^3} + \cdots \tag{5.3}$$

$$2^n x - \frac{1}{4} = 2^{n-1}a_1 + \cdots + a_n + \frac{a_{n+1}}{2} + \frac{a_{n+2}-1}{2^2} + \frac{a_{n+3}}{2^3} + \cdots \tag{5.4}$$

The difference between the integer parts is either 0 or 1. The condition is analyzed as

$$\lfloor 2^n x\rfloor - \left\lfloor 2^n x - \frac{1}{4}\right\rfloor = 1 \qquad \Longleftrightarrow \qquad \frac{a_{n+1}}{2} + \frac{a_{n+2}-1}{2^2} + \frac{a_{n+3}}{2^3} + \cdots < 0. \tag{5.5}$$

Therefore, we conclude

$$\lfloor 2^n x\rfloor - \left\lfloor 2^n x - \frac{1}{4}\right\rfloor = \begin{cases} 1, & a_{n+1} = a_{n+2} = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

Taking the maximum over $n \in \mathbb{N}$ simply picks out the earliest index n that satisfies $a_{n+1} = a_{n+2} = 0$. Define the following probability

$$P(X_n) = \text{Prob. of first having } a_{n+1} = a_{n+2} = 0 \text{ in an infinite binary string.}$$

We can write down

$$P(X_0) = \frac{1}{4}, \quad P(X_1) = \frac{1}{8}. \tag{5.7}$$

Note that event X_n indicates that $a_n = 1$. The recurrence relation can be obtained as

$$P(X_{n+2}) = \frac{1}{4}P(X_n) + \frac{1}{2}P(X_{n+1}). \tag{5.8}$$

The integral is now equivalent to an **expectation**

$$\begin{aligned} I &= \sum_{n=0}^{+\infty} \frac{P(X_n)}{2^n} = P(X_0) + \frac{1}{2}P(X_1) + \sum_{n=0}^{+\infty} \frac{1}{2^{n+2}} \cdot \frac{1}{4}P(X_n) + \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} \cdot \frac{1}{2}P(X_n) \\ &= \frac{1}{4} + \frac{1}{16} + \frac{I}{16} + \frac{1}{4}\left(I - \frac{1}{4}\right). \end{aligned} \tag{5.9}$$

Finally, the result is

$$I = \frac{1}{4} + \frac{5}{16}I, \qquad I = \frac{4}{11}. \tag{5.10}$$

Tiebreakers Question 1

$$\int \frac{\mathrm{d}x}{\sqrt[4]{x^4+1}} \tag{6.1}$$

Solution With several **changes of variables**, we have

$$\begin{aligned} I &= \int \frac{\mathrm{d}x}{\sqrt[4]{x^4+1}} \\ &= -\int \frac{1}{y}\frac{\mathrm{d}y}{\sqrt[4]{y^4+1}} \qquad \left(y=\frac{1}{x}, \quad \mathrm{d}x = -\frac{\mathrm{d}y}{y^2}\right) \\ &= -\int \frac{t^2}{t^4-1}\,\mathrm{d}t \qquad \left(t=\sqrt[4]{y^4+1}, \quad y=\sqrt[4]{t^4-1}, \quad \mathrm{d}y = t^3\left(t^4-1\right)^{-3/4}\,\mathrm{d}t\right). \end{aligned} \tag{6.2}$$

This leads to

$$\begin{aligned} I &= \frac{1}{2}\int \frac{\mathrm{d}t}{1-t^2} - \frac{1}{2}\int \frac{\mathrm{d}t}{1+t^2} \\ &= \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| - \frac{1}{2}\arctan t + C = \frac{1}{2}\operatorname{arctanh} t - \frac{1}{2}\arctan t + C. \end{aligned} \tag{6.3}$$

Going back to the original variable, we have

$$t = \frac{\sqrt[4]{x^4+1}}{x}, \qquad I = -\frac{1}{2}\arctan\left(\frac{\sqrt[4]{x^4+1}}{x}\right) + \frac{1}{4}\ln\left|\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x}\right| + C. \tag{6.4}$$

Note Using the identity $\arctan x + \arctan x^{-1} = \pi/2$, we can also write

$$I = \frac{1}{2}\arctan\left(\frac{x}{\sqrt[4]{x^4+1}}\right) + \frac{1}{4}\ln\left|\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x}\right| + C. \tag{6.5}$$

Tiebreakers Question 2

$$\int_0^{2\pi} \frac{(\sin 2x - 5\sin x)\sin x}{\cos 2x - 10\cos x + 13}\,\mathrm{d}x \tag{7.1}$$

Solution Note that

$$\begin{aligned} I &= \int_0^{2\pi} \frac{(\sin 2x - 5\sin x)\sin x}{\cos 2x - 10\cos x + 13}\,\mathrm{d}x = \int_0^{2\pi} \frac{(2\cos x - 5)\left(1-\cos^2 x\right)}{2(\cos x - 2)(\cos x - 3)}\,\mathrm{d}x \\ &= -\int_0^{2\pi}\left(\cos x + \frac{5}{2} + \frac{3}{2}\frac{1}{\cos x - 2} + \frac{4}{\cos x - 3}\right)\mathrm{d}x \\ &= -5\pi - \frac{3}{2}\int_0^{2\pi}\frac{\mathrm{d}x}{\cos x - 2} - 4\int_0^{2\pi}\frac{\mathrm{d}x}{\cos x - 3}. \end{aligned} \tag{7.2}$$

For the integral of **rational functions of cosine and sine**, we have the general result

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \oint_{|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{\mathrm{d}z}{iz}. \tag{7.3}$$

Therefore, we have

$$F(a) = \int_0^{2\pi}\frac{\mathrm{d}x}{\cos x + a} = -2i\oint_{|z|=1}\frac{\mathrm{d}z}{z^2 + 2az + 1}. \tag{7.4}$$

Using the **residue theorem**, we have

$$F(-2) = -\frac{2\pi}{\sqrt{3}}, \qquad F(-3) = -\frac{\pi}{\sqrt{2}}. \tag{7.5}$$

Finally, we obtain

$$I = -5\pi - \frac{3}{2}F(-2) - 4F(-3) = \left(\sqrt{3} + 2\sqrt{2} - 5\right)\pi. \tag{7.6}$$

Note The poles of function $f(z) = \left(z^2 + 2az + 1\right)^{-1}$ are

$$z_1 = -a - \sqrt{a^2 - 1}, \qquad z_2 = -a + \sqrt{a^2 - 1}. \tag{7.7}$$

Therefore, we need to discuss different cases to apply the residue theorem.

(a) When $a > 1$, we have $|z_1| > 1$ and $|z_2| < 1$.

$$F(a) = 4\pi\,\mathrm{Res}(f, z_2) = \frac{4\pi}{z_2 - z_1} = \frac{2\pi}{\sqrt{a^2-1}} \qquad \text{when } a > 1. \tag{7.8}$$

(b) When $a < -1$, we have $|z_1| < 1$ and $|z_2| > 1$.

$$F(a) = 4\pi\,\mathrm{Res}(f, z_1) = \frac{4\pi}{z_1 - z_2} = -\frac{2\pi}{\sqrt{a^2-1}} \qquad \text{when } a < -1. \tag{7.9}$$

(c) When $-1 \le a \le 1$, the integral does not converge.

Tiebreakers Question 3

$$\int \sqrt{x^4 - 4x + 3}\,\mathrm{d}x \tag{8.1}$$

Solution We first have

$$\begin{aligned} I &= \int (x-1)\sqrt{x^2+2x+3}\,\mathrm{d}x \\ &= \frac{1}{2}\int \sqrt{x^2+2x+3}\,\mathrm{d}\left(x^2+2x+3\right) - 2\int \sqrt{(x+1)^2+2}\,\mathrm{d}x. \end{aligned} \tag{8.2}$$

Note that

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C. \tag{8.3}$$

Therefore, we obtain

$$I = \frac{1}{3}\left(x^2+2x+3\right)^{\frac{3}{2}} - (x+1)\sqrt{x^2+2x+3} - 2\ln\left|x+1+\sqrt{x^2+2x+3}\right| + C. \tag{8.4}$$

Tiebreakers Question 4

$$\int_{-\infty}^{+\infty} \sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\left[4\cos^2\left(2^x\right)\,\left(4\cos^2\left(3^x\right)-3\right)^2-1\right]\mathrm{d}x \tag{9.1}$$

Solution Using the **triple-angle formula** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we have

$$\begin{aligned} I &= \int_{-\infty}^{+\infty}\left[4\sin^2\left(2^x\right)\,\cos^2\left(2^x\right)\,\cos^2\left(3^{x+1}\right)\,-\sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\right]\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}\left[\sin^2\left(2^{x+1}\right)\,\cos^2\left(3^{x+1}\right)-\sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\right]\,\mathrm{d}x. \end{aligned} \tag{9.2}$$

Now denote

$$\begin{aligned} I_N &= \int_{-N}^{N}\left[\sin^2\left(2^{x+1}\right)\,\cos^2\left(3^{x+1}\right)-\sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\right]\,\mathrm{d}x \\ &= \int_{N}^{N+1}\sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\mathrm{d}x - \int_{-N}^{-N+1}\sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\mathrm{d}x. \end{aligned} \tag{9.3}$$

The second part goes to 0 as $N \to +\infty$ according to the **mean value theorem**. For the first part, we can evaluate it as

$$\begin{aligned} A_N &= \int_{N}^{N+1}\sin^2\left(2^x\right)\,\cos^2\left(3^x\right)\mathrm{d}x \\ &= \frac{1}{4}+\frac{1}{4}\int_{N}^{N+1}\left[\cos\left(2\cdot 3^x\right)-\cos\left(2^{x+1}\right)-\cos\left(2\cdot 3^x\right)\cos\left(2^{x+1}\right)\right]\,\mathrm{d}x. \end{aligned} \tag{9.4}$$

Therefore, we conclude

$$I = \lim_{N\to+\infty} I_N = \lim_{N\to+\infty} A_N = \frac{1}{4}. \tag{9.5}$$

Tiebreakers Question 5

$$\int_2^\infty \frac{\lfloor x \rfloor x^2}{x^6 - 1}\,\mathrm{d}x \tag{10.1}$$

Solution Note that

$$\begin{aligned} I &= \int_2^\infty \frac{\lfloor x \rfloor x^2}{x^6 - 1}\,\mathrm{d}x = \sum_{n=2}^{+\infty} \int_n^{n+1} \frac{n x^2}{x^6 - 1}\,\mathrm{d}x \\ &= \sum_{n=2}^{+\infty} \frac{n}{3} \int_n^{n+1} \frac{\mathrm{d}x^3}{x^6 - 1} = \sum_{n=2}^{+\infty} \frac{n}{6} \ln\left(\frac{t^3 - 1}{t^3 + 1}\right)\Bigg|_n^{n+1}. \end{aligned} \tag{10.2}$$

Denote the following functions $g(n)$ and $h(n)$ for convenience

$$g(n) = \ln\left(\frac{n-1}{n+1}\right), \qquad h(n) = \ln\left(\frac{n^2 + n + 1}{n^2 - n + 1}\right) = \ln\left[\frac{n(n+1) + 1}{(n-1)n + 1}\right]. \tag{10.3}$$

Therefore, we have

$$f(n) = g(n) + h(n) = \ln\left(\frac{n^3 - 1}{n^3 + 1}\right) \tag{10.4}$$

and the integral becomes

$$\begin{aligned} I &= \frac{1}{6} \sum_{n=2}^{+\infty} \left[n f(n+1) - n f(n)\right] \\ &= \frac{1}{6}\left[\lim_{n\to+\infty} n f(n+1) - f(2) - \sum_{n=2}^{+\infty} f(n)\right]. \end{aligned} \tag{10.5}$$

We can first calculate the limit as

$$\lim_{n\to+\infty} n f(n+1) = \lim_{n\to+\infty} \frac{f(n)}{n^{-1}} = \lim_{n\to+\infty} -n^2 f'(n) = \lim_{n\to+\infty} \frac{6n^4}{n^6 - 1} = 0. \tag{10.6}$$

Furthermore, the infinite sum is

$$\sum_{n=2}^{+\infty} f(n) = \sum_{n=2}^{+\infty} \left[g(n) + h(n)\right] = \ln 2 - \ln 3. \tag{10.7}$$

Finally, we have

$$I = -\frac{1}{6}\left[f(2) + \ln 2 - \ln 3\right] = \frac{1}{6} \ln\left(\frac{27}{14}\right). \tag{10.8}$$

Lightning Question 1

$$\int_0^1 \left[\left(1 - x^{\frac{3}{2}}\right)^{\frac{3}{2}} - \left(1 - x^{\frac{2}{3}}\right)^{\frac{2}{3}}\right] \mathrm{d}x \tag{11.1}$$

Solution 1 Using the **Beta function**, we have

$$\begin{aligned} F(\alpha) &= \int_0^1 (1 - x^\alpha)^\alpha \, \mathrm{d}x \\ &= \frac{1}{\alpha} \int_0^1 t^{1/\alpha - 1} (1 - t)^\alpha \, \mathrm{d}t \qquad \left(t = x^\alpha, \quad \mathrm{d}x = \frac{t^{1/\alpha - 1}}{\alpha} \mathrm{d}t\right) \\ &= \frac{1}{\alpha} B\left(\alpha^{-1}, \alpha + 1\right) = \frac{\Gamma(\alpha)\, \Gamma\left(\alpha^{-1}\right)}{\Gamma\left(\alpha + \alpha^{-1} + 1\right)}. \end{aligned} \tag{11.2}$$

The property $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ is applied. Therefore, we obtain a more general result

$$I = F(\alpha) - F(\alpha^{-1}) = 0 \qquad \text{for } \alpha > 0. \tag{11.3}$$

Solution 2 Note that the two functions are inverses of each other

$$y = f(x) = (1 - x^\alpha)^\alpha, \qquad x = f^{-1}(y) = \left(1 - y^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}. \tag{11.4}$$

Therefore, using **integration by parts**, because $y(0) = 1$ and $y(1) = 0$, we have

$$F(\alpha) = \int_0^1 y \, \mathrm{d}x = xy \Big|_{x=0}^{x=1} - \int_{y(0)}^{y(1)} x(y) \, \mathrm{d}y = \int_0^1 x \, \mathrm{d}y = F(\alpha^{-1}). \tag{11.5}$$

Lightning Question 2

$$\int \left(\frac{x}{x-1}\right)^4 \mathrm{d}x \tag{12.1}$$

Solution

$$\begin{aligned} I &= \int \left(1 + \frac{1}{x-1}\right)^4 \mathrm{d}x \\ &= x + 4\ln|x-1| - \frac{6}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{3(x-1)^3} + C. \end{aligned} \tag{12.2}$$

Lightning Question 3

$$\int \frac{[\tan(1012x) + \tan(1013x)]\cos(1012x)\cos(1013x)}{\cos(2025x)} \mathrm{d}x \tag{13.1}$$

Solution

$$\begin{aligned} I &= \int \frac{\sin(1012x)\cos(1013x) + \sin(1013x)\cos(1012x)}{\cos(2025x)} \mathrm{d}x \\ &= \int \tan(2025x)\,\mathrm{d}x = -\frac{1}{2025}\ln|\cos(2025x)| + C. \end{aligned} \tag{13.2}$$