

MIT Integration Bee: 2023 Semifinal

Semifinal #1

Question 1

$$\int e^{\cos x} \cos(2x + \sin x) dx \quad (1.1)$$

Solution

$$\begin{aligned} I &= \int e^{\cos x} \cos x \cos(x + \sin x) dx + \int \sin(x + \sin x) d(e^{\cos x}) \\ &= e^{\cos x} \sin(x + \sin x) - \int e^{\cos x} \cos(x + \sin x) dx \\ &= e^{\cos x} \sin(x + \sin x) - \int e^{\cos x} \cos x \cos(\sin x) dx - \int \sin(\sin x) d(e^{\cos x}) \\ &= e^{\cos x} \sin(x + \sin x) - e^{\cos x} \sin(\sin x) + C. \end{aligned} \quad (1.2)$$

Question 2

$$\int_0^1 (9x^9 - x^{90} + 9x^{99} - x^{900} + 9x^{909} - x^{990} + 9x^{999} - x^{9000} + \dots) dx \quad (2.1)$$

Solution The contribution from the dominant terms in the integrand is

$$I_1 = \int_0^1 (9x^9 + 9x^{99} + 9x^{999} + 9x^{9999} + \dots) dx = \sum_{k=1}^{\infty} \frac{9}{10^k} = 1. \quad (2.2)$$

The remaining part becomes

$$\begin{aligned} I_2 &= \int_0^1 (-x^{90} - x^{900} + 9x^{909} - x^{990} \\ &\quad - x^{9000} + 9x^{9009} - x^{9090} + 9x^{9099} - x^{9900} + 9x^{9909} - x^{9990} - \dots) dx. \end{aligned} \quad (2.3)$$

This can be considered an alternating series that eventually sums to zero. Therefore, we can say

$$I = I_1 + I_2 = 1. \quad (2.4)$$

Question 3

$$\int_0^\pi \frac{2 \cos x - \cos(2021x) - 2 \cos(2022x) - \cos(2023x) + 2}{1 - \cos 2x} dx \quad (3.1)$$

Solution Using the trigonometric identity, we have

$$\cos(2021x) + \cos(2023x) = 2 \cos(2022x) \cos x. \quad (3.2)$$

Therefore, the integral becomes

$$I = \int_0^\pi \frac{2(1 + \cos x)[1 - \cos(2022x)]}{2(1 - \cos^2 x)} dx = \int_0^\pi \frac{1 - \cos(2022x)}{1 - \cos x} dx. \quad (3.3)$$

Denote the general result as

$$I_n = \int_0^\pi \frac{1 - \cos nx}{1 - \cos x} dx, \quad n \in \mathbb{N}. \quad (3.4)$$

Obviously we have $I_0 = 0$ and $I_1 = \pi$. The **reduction formula** can be obtained as

$$\begin{aligned} I_{n+1} &= \int_0^\pi \frac{1 - \cos(n+1)x}{1 - \cos x} dx = \int_0^\pi \frac{1 + \cos(n-1)x - 2 \cos nx \cos x}{1 - \cos x} dx \\ &= -I_{n-1} + \int_0^\pi \frac{2 - 2 \cos nx + 2 \cos nx - 2 \cos nx \cos x}{1 - \cos x} dx \\ &= 2I_n - I_{n-1} + \int_0^\pi 2 \cos nx dx = 2I_n - I_{n-1} \quad \text{for } n \geq 1. \end{aligned} \quad (3.5)$$

This implies that $\{I_n\}$ is an arithmetic sequence, and thus $I_n = n\pi$. Finally, we have

$$I = I_{2022} = 2022\pi. \quad (3.6)$$

Question 4

$$\int \frac{3 \ln x - 1 + 2x}{x \ln x + x^2 + 2x^4} dx \quad (4.1)$$

Solution Note that

$$\frac{d}{dx} \ln(\ln x + x + 2x^3) = \frac{1 + x + 6x^3}{x(\ln x + x + 2x^3)}, \quad \frac{1}{x} = \frac{\ln x + x + 2x^3}{x(\ln x + x + 2x^3)}. \quad (4.2)$$

The integrand can be written as

$$\frac{3 \ln x - 1 + 2x}{x \ln x + x^2 + 2x^4} = \frac{3}{x} - \frac{1 + x + 6x^3}{x(\ln x + x + 2x^3)}. \quad (4.3)$$

Therefore, the integral is calculated as

$$I = 3 \ln x - \ln(\ln x + x + 2x^3) + C. \quad (4.4)$$

Tiebreakers Question 1

$$\int_0^{\ln 2} \left\{ \frac{1}{e^x - 1} \right\} dx \quad (5.1)$$

Solution With the **change of variable**, we have

$$t = \frac{1}{e^x - 1}, \quad x = \ln \left(\frac{t+1}{t} \right), \quad dx = -\frac{dt}{t(t+1)}. \quad (5.2)$$

Therefore, the integral becomes

$$I = \int_1^{+\infty} \frac{\{t\} dt}{t(t+1)} = \sum_{n=1}^{+\infty} \int_0^1 \frac{u du}{(u+n)(u+n+1)}. \quad (5.3)$$

Each single term can be evaluated as

$$\begin{aligned} \int_0^1 \frac{u du}{(u+n)(u+n+1)} &= \int_0^1 \left(\frac{n+1}{u+n+1} - \frac{n}{u+n} \right) du \\ &= (n+1) [\ln(n+2) - \ln(n+1)] - n [\ln(n+1) - \ln n] \\ &= (n+1) \ln(n+2) - n \ln(n+1) + n \ln n - (n+1) \ln(n+1). \end{aligned} \quad (5.4)$$

Now denote the following two series

$$a_n = n \ln(n+1), \quad b_n = n \ln n, \quad n = 1, 2, 3, \dots \quad (5.5)$$

The integral can thus be calculated as

$$\begin{aligned} I &= \sum_{n=1}^{+\infty} (a_{n+1} - a_n + b_n - b_{n+1}) = b_1 - a_1 + \lim_{n \rightarrow \infty} (a_n - b_n) \\ &= -\ln 2 + \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n} \right)^n = 1 - \ln 2. \end{aligned} \quad (5.6)$$

Tiebreakers Question 2

$$\int_1^9 \left(\sqrt[3]{x + \sqrt{x^2 - 1}} + \sqrt[3]{x - \sqrt{x^2 - 1}} \right) dx \quad (6.1)$$

Solution Similar approach to [2024 Final: Question 4](#). Denote

$$a(x) = \sqrt[3]{x + \sqrt{x^2 - 1}}, \quad b(x) = \sqrt[3]{x - \sqrt{x^2 - 1}}, \quad y(x) = a(x) + b(x). \quad (6.2)$$

Now we can show that

$$a^3 + b^3 = 2x, \quad ab = 1 \quad \implies \quad y^3 = a^3 + b^3 + 3ab(a + b) = 2x + 3y. \quad (6.3)$$

Using **integration by parts**, we obtain

$$I = \int_1^9 y(x) dx = xy \Big|_{x=1}^{x=9} - \int_{y(1)}^{y(9)} x(y) dy. \quad (6.4)$$

Since we have

$$y^3 = 2x + 3y \quad \implies \quad y(1) = 2, \quad y(9) = 3, \quad x(y) = \frac{1}{2}y^3 - \frac{3}{2}y, \quad (6.5)$$

we finally obtain

$$I = 25 - \frac{1}{2} \int_2^3 (y^3 - 3y) dy = 25 - \frac{35}{8} = \frac{165}{8}. \quad (6.6)$$

Semifinal #2

Question 1

$$\int (\sqrt{x+1} - \sqrt{x})^\pi dx \quad (7.1)$$

Solution Denote $y = \sqrt{x+1} - \sqrt{x}$, and then we can show that

$$y^2 - y^{-2} = (\sqrt{x+1} - \sqrt{x})^2 - (\sqrt{x+1} + \sqrt{x})^2 = -4\sqrt{x}\sqrt{x+1}. \quad (7.2)$$

The derivative y' can thus be expressed as

$$y' = \frac{\sqrt{x} - \sqrt{x+1}}{2\sqrt{x}\sqrt{x+1}} = \frac{2y}{y^2 - y^{-2}}. \quad (7.3)$$

Hence, we can obtain

$$\frac{d}{dx} \left(\frac{y^{n+1}}{n+1} \right) = y^n y' = \frac{2y^{n+1}}{y^2 - y^{-2}}. \quad (7.4)$$

Based on Eq. (7.4), when $n \neq \pm 2$ we have

$$\frac{d}{dx} \left(\frac{y^{n+2}}{n+2} - \frac{y^{n-2}}{n-2} \right) = \frac{2(y^{n+2} - y^{n-2})}{y^2 - y^{-2}} = 2y^n. \quad (7.5)$$

Now, the original integral is calculated as

$$I = \frac{1}{2} \left[\frac{(\sqrt{x+1} - \sqrt{x})^{\pi+2}}{\pi+2} - \frac{(\sqrt{x+1} - \sqrt{x})^{\pi-2}}{\pi-2} \right] + C. \quad (7.6)$$

Question 2

$$\int_{-2}^2 (((x^2 - 2)^2 - 2)^2 - 2) dx \quad (8.1)$$

Solution With the **change of variable** $x = 2 \cos \theta$ with $\theta \in [0, \pi]$, we have

$$x^2 - 2 = 4 \cos^2 \theta - 2 = 2 \cos 2\theta, \quad dx = -2 \sin \theta d\theta. \quad (8.2)$$

Then we know that

$$f(x) = x^2 - 2 = 2 \cos 2\theta, \quad f^2(x) = (f \circ f)(x) = 2 \cos 4\theta, \quad f^4(x) = 2 \cos 16\theta. \quad (8.3)$$

Therefore, the integral is evaluated as

$$I = \int_0^\pi 4 \cos 16\theta \sin \theta d\theta = 2 \int_0^\pi (\sin 17\theta - \sin 15\theta) d\theta = \frac{4}{17} - \frac{4}{15} = -\frac{8}{255}. \quad (8.4)$$

Question 3

$$\int_0^{+\infty} \frac{\tanh x}{x \cosh 2x} dx \quad (9.1)$$

Solution The integrand can be written as

$$\begin{aligned} \frac{\tanh x}{x \cosh 2x} &= \frac{1}{x} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{2}{e^{2x} + e^{-2x}} \\ &= \frac{2}{x} \cdot \frac{e^{2x} (e^{2x} - 1)}{(e^{2x} + 1)(e^{4x} + 1)} = \frac{2}{x} \left(\frac{1}{e^{2x} + 1} - \frac{1}{e^{4x} + 1} \right). \end{aligned} \quad (9.2)$$

For the **Frullani integrals**, we have the following property

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} dx = [f(\infty) - f(0)] \ln \frac{a}{b}. \quad (9.3)$$

The integral can now be written in the form of

$$f(x) = \frac{2}{e^x + 1}, \quad I = \int_0^{+\infty} \frac{f(2x) - f(4x)}{x} dx. \quad (9.4)$$

Finally, we can evaluate it as

$$I = [f(\infty) - f(0)] \ln \frac{1}{2} = \ln 2. \quad (9.5)$$

Question 4

$$\int \sin(4 \arctan x) dx \quad (10.1)$$

Solution With the **change of variable** $t = \arctan x$, we have

$$\tan t = x, \quad \sin t = \frac{x}{\sqrt{x^2 + 1}}, \quad \cos t = \frac{1}{\sqrt{x^2 + 1}}, \quad dx = \frac{dt}{\cos^2 t}. \quad (10.2)$$

The integral can thus be computed as

$$\begin{aligned} I &= \int \frac{\sin 4t}{\cos^2 t} dt = \int \frac{2 \sin 2t (2 \cos^2 t - 1)}{\cos^2 t} dt \\ &= \int (4 \sin 2t - 4 \tan t) dt = -2 \cos 2t - 4 \ln |\sec t| + C. \end{aligned} \quad (10.3)$$

We also know that

$$\cos 2t = \frac{1 - x^2}{x^2 + 1} = -1 + \frac{2}{x^2 + 1}, \quad \sec t = \sqrt{x^2 + 1}. \quad (10.4)$$

Therefore, the final result becomes

$$I = -\frac{4}{x^2 + 1} - 2 \ln(x^2 + 1) + C. \quad (10.5)$$