MIT Integration Bee: 2023 Regular Season

Question 1

$$\int_0^{2\pi} \max\{\sin x, \sin 2x\}\,\mathrm{d}x \tag{1.1}$$

Solution

$$\begin{aligned} I &= \int_0^{\pi/3} \sin 2x\,\mathrm{d}x + \int_{\pi/3}^{\pi} \sin x\,\mathrm{d}x + \int_{\pi}^{5\pi/3} \sin 2x\,\mathrm{d}x + \int_{5\pi/3}^{2\pi} \sin x\,\mathrm{d}x \\ &= 2\int_0^{\pi/3} \sin 2x\,\mathrm{d}x + \int_{\pi/3}^{2\pi/3} \sin x\,\mathrm{d}x = 2 \times \frac{3}{4} + 1 = \frac{5}{2}. \end{aligned} \tag{1.2}$$

Question 2

$$\int_0^1 \frac{x+1/x+2}{x+3/x+4}\,\mathrm{d}x \tag{2.1}$$

Solution

$$I = \int_0^1 \frac{x^2+5x+4}{x^2+5x+6}\,\mathrm{d}x = 1 - 2\int_0^1 \left(\frac{1}{x+2} - \frac{1}{x+3}\right)\mathrm{d}x = 1 - \ln\left(\frac{81}{64}\right). \tag{2.2}$$

Question 3

$$\int_0^3 \left[\min\left\{2x, \frac{5-x}{2}\right\} - \max\left\{-\frac{x}{2}, 2x-5\right\}\right]\mathrm{d}x \tag{3.1}$$

Solution

$$I = \int_0^1 2x\,\mathrm{d}x + \int_1^3 \frac{5-x}{2}\,\mathrm{d}x - \int_0^2 -\frac{x}{2}\,\mathrm{d}x - \int_2^3 (2x-5)\,\mathrm{d}x = 1+3+1+0 = 5. \tag{3.2}$$

Question 4

$$\int 1 - \underbrace{\frac{1}{1 - \frac{1}{\ddots \frac{1}{1-\frac{1}{x}}}}}_{2023\ (1-)\text{'s}} \,\mathrm{d}x \tag{4.1}$$

Solution Denote the following series

$$a_1 = 1 - \frac{1}{x}, \qquad a_n = 1 - \frac{1}{a_{n-1}} \qquad \text{for } n \geq 2. \tag{4.2}$$

We can show that $\{a_n\}$ is 3-periodic

$$a_1 = 1 - \frac{1}{x}, \qquad a_2 = \frac{1}{1-x}, \qquad a_3 = x, \qquad a_4 = 1 - \frac{1}{x} = a_1. \tag{4.3}$$

Therefore, the integral is computed as

$$I = \int 1 - \frac{1}{x} \,\mathrm{d}x = x - \ln x + C. \tag{4.4}$$

Question 5

$$\int_0^{\pi/2} x \cot x \,\mathrm{d}x \tag{5.1}$$

Solution

$$I = x \ln(\sin x) \Big|_0^{\pi/2} - \int_0^{\pi/2} \ln(\sin x) \,\mathrm{d}x = -\int_0^{\pi/2} \ln(\sin x) \,\mathrm{d}x. \tag{5.2}$$

This is a classic definite integral, and it is evaluated as follows

$$\begin{aligned} \int_0^{\pi/2} \ln(\sin x) \,\mathrm{d}x &= \frac{1}{2}\left[\int_0^{\pi/2} \ln(\sin x) \,\mathrm{d}x + \int_0^{\pi/2} \ln(\cos x) \,\mathrm{d}x\right] \\ &= \frac{1}{2}\int_0^{\pi/2} \ln(\sin 2x) \,\mathrm{d}x - \frac{\pi}{4}\ln 2 \\ &= \frac{1}{4}\int_0^{\pi} \ln(\sin t) \,\mathrm{d}t - \frac{\pi}{4}\ln 2 \\ &= \frac{1}{2}\int_0^{\pi/2} \ln(\sin t) \,\mathrm{d}t - \frac{\pi}{4}\ln 2. \end{aligned} \tag{5.3}$$

Therefore, we obtain

$$\int_0^{\pi/2} \ln(\sin x) \,\mathrm{d}x = -\frac{\pi}{2}\ln 2, \qquad I = \frac{\pi}{2}\ln 2. \tag{5.4}$$

Question 6

$$\int \left(\frac{x^6 + x^4 - x^2 - 1}{x^4}\right) e^{x+1/x}\,\mathrm{d}x \tag{6.1}$$

Solution Denote $f(x) = x + x^{-1}$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{x+1/x} = \left(1 - \frac{1}{x^2}\right) e^{x+1/x}. \tag{6.2}$$

The integral can be calculated as

$$\begin{aligned} I &= \int \left(x + \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2}\right) e^{x+1/x}\,\mathrm{d}x = \int [f(x)]^2 f'(x)\, e^{f(x)}\,\mathrm{d}x \\ &= \left\{[f(x)]^2 - 2f(x) + 2\right\} e^{f(x)} + C \\ &= \left(x^2 - 2x + 4 - \frac{2}{x} + \frac{1}{x^2}\right) e^{x+1/x} + C. \end{aligned} \tag{6.3}$$

Question 7

$$\int \frac{\mathrm{d}x}{\sqrt{(x+1)^3(x-1)}} \tag{7.1}$$

Solution Note that

$$\begin{aligned} \frac{1}{\sqrt{(x+1)^3(x-1)}} &= \frac{1}{\sqrt{(x+1)^2}} \cdot \frac{1}{\sqrt{(x+1)(x-1)}} \\ &= \frac{1}{\sqrt{(x+1)^2}} \cdot \frac{1}{2}\left(\sqrt{\frac{x+1}{x-1}} - \sqrt{\frac{x-1}{x+1}}\right). \end{aligned} \tag{7.2}$$

With $u = \sqrt{x-1}$ and $v = \sqrt{x+1}$, we can express the integrand as

$$\frac{1}{\sqrt{(x+1)^3(x-1)}} = \frac{u'v - uv'}{v^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right). \tag{7.3}$$

Therefore, the integral is calculated as

$$I = \sqrt{\frac{x-1}{x+1}} + C. \tag{7.4}$$

Question 8

$$\int_0^{\pi} x \sin^4 x \, dx \tag{8.1}$$

Solution

$$I = \frac{1}{4}\int_0^{\pi} x\,(1-\cos 2x)^2 \, dx = \frac{1}{8}\int_0^{\pi} x\,(3 - 4\cos 2x + \cos 4x) \, dx. \tag{8.2}$$

Since we have

$$I_n = \int_0^{\pi} x \cos nx \, dx = -\frac{1}{n}\int_0^{\pi} \sin nx \, dx = \frac{(-1)^n - 1}{n^2}, \tag{8.3}$$

the integral is evaluated as

$$I = \frac{1}{8}\left(\frac{3\pi^2}{2} - 0 + 0\right) = \frac{3\pi^2}{16}. \tag{8.4}$$

Question 9

$$\int \binom{x}{5}^{-1} dx \tag{9.1}$$

Solution

$$I = 120 \int \frac{dx}{x(x-1)(x-2)(x-3)(x-4)}. \tag{9.2}$$

The coefficients of the **partial fraction decomposition** can be obtained by taking the limits.

$$\frac{1}{x(x-1)(x-2)(x-3)(x-4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3} + \frac{E}{x-4}. \tag{9.3}$$

We can show that

$$A = \lim_{x\to 0} \frac{1}{(x-1)(x-2)(x-3)(x-4)} = 24. \tag{9.4}$$

Similarly, the coefficients are solved as

$$A = E = \frac{1}{24}, \qquad B = D = -\frac{1}{6}, \qquad C = \frac{1}{4}. \tag{9.5}$$

Hence, the integral can be computed as

$$I = 5\ln|x| - 20\ln|x-1| + 30\ln|x-2| - 20\ln|x-3| + 5\ln|x-4| + C. \tag{9.6}$$

Question 10

$$\int \frac{\sin 2x \cos 3x}{\sin^2 x \cos^3 x} \, \mathrm{d}x \tag{10.1}$$

Solution

$$I = \int \left(8 \cot x - \frac{6}{\sin x \cos x}\right) \mathrm{d}x = 8 \ln|\sin x| - 6 \ln|\tan x| + C. \tag{10.2}$$

Question 11

$$\int \left(\sqrt{2 \ln x} + \frac{1}{\sqrt{2 \ln x}}\right) \mathrm{d}x \tag{11.1}$$

Solution Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sqrt{\ln x}\right) = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}, \qquad I = x\sqrt{2 \ln x} + C. \tag{11.2}$$

Question 12

$$\int \frac{\ln(\cos x)}{\cos^2 x} \, \mathrm{d}x \tag{12.1}$$

Solution

$$I = \tan x \ln(\cos x) + \int \frac{\sin^2 x}{\cos^2 x} \, \mathrm{d}x = \tan x \ln(\cos x) + \tan x - x + C. \tag{12.2}$$

Question 13

$$\int_0^{\frac{\pi}{2}+1} \sin(x - \sin(x - \sin(x - \cdots))) \, \mathrm{d}x \tag{13.1}$$

Solution Denote the function as $y = y(x)$, and then we have

$$y = \sin(x - y), \qquad x = y + \arcsin y, \qquad y(0) = 0, \qquad y\left(\frac{\pi}{2} + 1\right) = 1. \tag{13.2}$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}+1} y \, \mathrm{d}x = \int_0^1 y \left(1 + \frac{1}{\sqrt{1 - y^2}}\right) \mathrm{d}y = \frac{1}{2} + 1 = \frac{3}{2}. \tag{13.3}$$

Question 14

$$\int_0^{100} \lfloor x \rfloor x \lceil x \rceil \, \mathrm{d}x \tag{14.1}$$

Solution With $N = 100$, the integral is evaluated as

$$\begin{aligned} I &= \sum_{k=1}^{100} \int_{k-1}^{k} k(k-1)x \, \mathrm{d}x = \frac{1}{2} \sum_{k=1}^{100} k(k-1)(2k-1) = \sum_{k=1}^{100} \left(k^3 - \frac{3}{2}k^2 + \frac{k}{2}\right) \\ &= \frac{N^2(N+1)^2}{4} - \frac{N(N+1)(2N+1)}{4} + \frac{N(N+1)}{4} = \frac{N^4 - N^2}{4} = \frac{100^4 - 100^2}{4}. \end{aligned} \tag{14.2}$$

Question 15

$$\int_{-\infty}^{+\infty} \frac{\frac{1}{(x-1)^2} + \frac{3}{(x-3)^4} + \frac{5}{(x-5)^6}}{1 + \left(\frac{1}{x-1} + \frac{1}{(x-3)^3} + \frac{1}{(x-5)^5}\right)^2} \, \mathrm{d}x \tag{15.1}$$

Solution Denote the following function and the integral can be expressed as

$$f(x) = \frac{1}{x-1} + \frac{1}{(x-3)^3} + \frac{1}{(x-5)^5}, \qquad I = \int_{-\infty}^{+\infty} \frac{-f'(x)}{1 + [f(x)]^2} \, \mathrm{d}x. \tag{15.2}$$

Note that there are **singularities** $x = 1, 3, 5$ after the substitution (see 2024 Semifinal #1: Question 1). The limits are analyzed as follows

$$\begin{aligned} &f(-\infty) \to 0, \quad f(1^-) \to -\infty, \quad f(1^+) \to +\infty, \quad f(3^-) \to -\infty, \\ &f(3^+) \to +\infty, \quad f(5^-) \to -\infty, \quad f(5^+) \to +\infty, \quad f(+\infty) \to 0. \end{aligned}$$

Therefore, the integral is evaluated as

$$\begin{aligned} I &= -\left(\arctan f \Big|_{x=-\infty}^{x=1^-} + \arctan f \Big|_{x=1^+}^{x=3^-} + \arctan f \Big|_{x=3^+}^{x=5^-} + \arctan f \Big|_{x=5^+}^{x=+\infty}\right) \\ &= -\left[\arctan(-\infty) - 0 + 2\arctan(-\infty) - 2\arctan(+\infty) + 0 - \arctan(+\infty)\right] \\ &= 6\arctan(+\infty) = 3\pi. \end{aligned} \tag{15.3}$$

Question 16

$$\int_0^{\pi} \sin^2\left(3x + \cos^4(5x)\right) \mathrm{d}x \tag{16.1}$$

Solution

$$I = \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi} \cos\left(6x + 2\cos^4(5x)\right) \mathrm{d}x = \frac{\pi}{2}. \tag{16.2}$$

Question 17

$$\int_0^{5} (-1)^{\lfloor x \rfloor + \lfloor x/\sqrt{2} \rfloor + \lfloor x/\sqrt{3} \rfloor} \mathrm{d}x \tag{17.1}$$

Solution The nodes within the interval $[0, 5]$ are listed below

$$0, \quad 1, \quad \sqrt{2}, \quad \sqrt{3}, \quad 2, \quad 2\sqrt{2}, \quad 3, \quad 2\sqrt{3}, \quad 4, \quad 3\sqrt{2}, \quad 5.$$

We need to add or subtract the length of each interval accordingly. The integral can be evaluated as

$$\begin{aligned} I = {} & -0 + 2 \times 1 - 2 \times \sqrt{2} + 2 \times \sqrt{3} - 2 \times 2 + 2 \times 2\sqrt{2} \\ & - 2 \times 3 + 2 \times 2\sqrt{3} - 2 \times 4 + 2 \times 3\sqrt{2} - 5 = 8\sqrt{2} + 6\sqrt{3} - 21. \end{aligned} \tag{17.2}$$

Question 18

$$\int_0^{+\infty} \frac{x+1}{x+2} \cdot \frac{x+3}{x+4} \cdot \frac{x+5}{x+6} \cdot \cdots \mathrm{d}x \tag{18.1}$$

Solution This integral should be 0, as the integrand always converges to 0 for all $x \in \mathbb{R}^+$. In other words, $f(x) \equiv 0$ for $x > 0$.

Question 19

$$\int_0^{\pi/2} \frac{\sin 23x}{\sin x}\,\mathrm{d}x \tag{19.1}$$

Solution We can obtain the **reduction formula**

$$\begin{aligned}
f_n(x) = \frac{\sin(nx)}{\sin x} &= \frac{\sin[(n-1)x]\cos x}{\sin x} + \cos[(n-1)x] \\
&= \frac{\sin[(n-2)x]\cos^2 x + \cos[(n-2)x]\cos x \sin x}{\sin x} + \cos[(n-1)x] \\
&= f_{n-2}(x) - \sin[(n-2)x]\sin x + \cos[(n-2)x]\cos x + \cos[(n-1)x] \\
&= f_{n-2}(x) + 2\cos[(n-1)x].
\end{aligned} \tag{19.2}$$

Therefore, the corresponding definite integrals satisfy

$$I_n = I_{n-2} + 2\int_0^{\pi/2} \cos[(n-1)x]\,\mathrm{d}x = \begin{cases} I_{n-2} & n \text{ is odd,} \\ I_{n-2} + \dfrac{2}{n-1}\sin\left(\dfrac{n-1}{2}\pi\right) & n \text{ is even.} \end{cases} \tag{19.3}$$

The original problem can thus be evaluated as

$$I_{23} = I_1 = \frac{\pi}{2}. \tag{19.4}$$

Question 20

$$\int_1^{100} \left(\frac{\lfloor x/2 \rfloor}{\lfloor x \rfloor} + \frac{\lceil x/2 \rceil}{\lceil x \rceil}\right)\mathrm{d}x \tag{20.1}$$

Solution We first obtain

$$i_{2k} = \int_{2k}^{2k+1} \left(\frac{\lfloor x/2 \rfloor}{\lfloor x \rfloor} + \frac{\lceil x/2 \rceil}{\lceil x \rceil}\right)\mathrm{d}x = \int_{2k}^{2k+1} \left(\frac{k}{2k} + \frac{k+1}{2k+1}\right)\mathrm{d}x = \frac{1}{2} + \frac{k+1}{2k+1}, \tag{20.2}$$

$$i_{2k+1} = \int_{2k+1}^{2k+2} \left(\frac{\lfloor x/2 \rfloor}{\lfloor x \rfloor} + \frac{\lceil x/2 \rceil}{\lceil x \rceil}\right)\mathrm{d}x = \int_{2k+1}^{2k+2} \left(\frac{k}{2k+1} + \frac{k+1}{2k+2}\right)\mathrm{d}x = \frac{k}{2k+1} + \frac{1}{2}. \tag{20.3}$$

Therefore, we conclude that

$$i_{2k} + i_{2k+1} = 2, \qquad \forall\, k \in \mathbb{N}. \tag{20.4}$$

The original problem can thus be evaluated as

$$I = \sum_{n=1}^{99} i_n = 100 - i_0 = 100 - \frac{3}{2} = \frac{197}{2}. \tag{20.5}$$