# MIT Integration Bee: 2023 Quarterfinal

## Quarterfinal #1

### Question 1

$$\int_0^1 \frac{x^4 (1-x)^2}{1+x^2}\, \mathrm{d}x \tag{1.1}$$

**Solution**

$$I = \int_0^1 \left(x^4 - 2x^3 + 2x\right) \mathrm{d}x - \int_0^1 \frac{2x}{1+x^2}\, \mathrm{d}x = \frac{7}{10} - \ln 2. \tag{1.2}$$

### Question 2

$$\int \left( \begin{array}{r} \cos(3x)\cos(5x)\cos(6x)\cos(7x) \\ - \cos(x)\cos(2x)\cos(4x)\cos(8x) \end{array} \right) \mathrm{d}x \tag{2.1}$$

**Solution** Repeatedly using the trigonometric identity (**product-to-sum**), we have

$$\begin{aligned} I &= \frac{1}{8} \int \left( \begin{array}{r} \cos\{21x, 15x, 11x, 9x, 7x, 5x, 3x, x\} \\ - \cos\{15x, 13x, 11x, 9x, 7x, 5x, 3x, x\} \end{array} \right) \mathrm{d}x \\ &= \frac{1}{8} \int (\cos 21x - \cos 13x)\, \mathrm{d}x \\ &= \frac{1}{8} \left( \frac{\sin 21x}{21} - \frac{\sin 13x}{13} \right) + C. \end{aligned} \tag{2.2}$$

# Quarterfinal #2

## Question 1

$$\int_{\sqrt{e}}^{+\infty} x^{-\ln x}\,\mathrm{d}x \tag{3.1}$$

**Solution** With the **change of variable** $t = \ln x$, we have

$$I = \int_{1/2}^{+\infty} e^{-t^2+t}\,\mathrm{d}t = e^{1/4}\int_0^{+\infty} e^{-u^2}\,\mathrm{d}u = e^{1/4}\,\frac{\sqrt{\pi}}{2}. \tag{3.2}$$

## Question 2

$$\int \frac{1-2x}{(1+x)^2\,x^{2/3}}\,\mathrm{d}x \tag{4.1}$$

**Solution**

$$\begin{aligned} I &= \int \frac{1+x-3x}{(1+x)^2\,x^{2/3}}\,\mathrm{d}x = \int \frac{1}{(1+x)\,x^{2/3}}\,\mathrm{d}x - \int \frac{3x^{1/3}}{(1+x)^2}\,\mathrm{d}x \\ &= \frac{3x^{1/3}}{1+x} - \int 3x^{1/3}\,\mathrm{d}\left(\frac{1}{1+x}\right) - \int \frac{3x^{1/3}}{(1+x)^2}\,\mathrm{d}x = \frac{3x^{1/3}}{1+x} + C. \end{aligned} \tag{4.2}$$

## Question 3

$$\lim_{n\to\infty}\left[\frac{1}{n}\int_0^n \cos^2\left(\frac{\pi x^2}{\sqrt{2}}\right)\mathrm{d}x\right] \tag{5.1}$$

**Solution**

$$I = \frac{1}{2} + \lim_{n\to\infty}\left[\frac{1}{n}\int_0^n \cos\left(\sqrt{2}\pi x^2\right)\mathrm{d}x\right] = \frac{1}{2}. \tag{5.2}$$

Note that the **Fresnel integral** converges to a finite value as $n \to \infty$, which implies that the average over the interval tends to zero.

# Quarterfinal #3

## Question 1

$$\int_0^{2^{10}} \sum_{n=0}^{\infty} \left\{ \frac{x}{2^n} \right\} \mathrm{d}x \tag{6.1}$$

**Solution** The integral is evaluated as the sum of many elementary triangles, which is expressed as

$$\begin{aligned} I &= \sum_{n=0}^{10} 2^{10-n} \cdot 2^{n-1} + \sum_{n=11}^{\infty} 2^{10} \cdot 2^{10-n-1} \\ &= 11 \times 2^9 + \sum_{n=-\infty}^{8} 2^n = 12 \times 2^9 = 6144. \end{aligned} \tag{6.2}$$

## Question 2

$$\int_0^{+\infty} \operatorname{sech}^2 (x + \tan x) \, \mathrm{d}x \tag{7.1}$$

**Solution** Based on the **Glasser's master theorem** (see 2024 Semifinal #1: Tiebreaker 2), the integral is equivalent to

$$I = \frac{1}{2} \int_{-\infty}^{+\infty} \operatorname{sech}^2 x \, \mathrm{d}x = \frac{1}{2} \tanh x \Big|_{-\infty}^{+\infty} = 1. \tag{7.2}$$

**Note** This is because $\tan z$ has the following pole expansion

$$\tan x = -\sum_{n=0}^{+\infty} \left[ \frac{1}{z - (2n+1)\pi/2} + \frac{1}{z + (2n+1)\pi/2} \right]. \tag{7.3}$$

Another classic example is

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{+\infty} \left[ \frac{1}{z-n} + \frac{1}{z+n} \right]. \tag{7.4}$$

# Quarterfinal #4

## Question 1

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{1+\cos x+\sin x} \tag{8.1}$$

**Solution** With the **universal change of variable**, we have

$$t=\tan\frac{x}{2}\in[0,1], \qquad \cos x=\frac{1-t^2}{1+t^2}, \qquad \sin x=\frac{2t}{1+t^2}, \qquad \mathrm{d}x=\frac{2\,\mathrm{d}t}{1+t^2}. \tag{8.2}$$

The integral can thus be computed as

$$I=\int_0^1 \frac{2\,\mathrm{d}t}{(1+t^2)+(1-t^2)+2t}=\int_0^1 \frac{\mathrm{d}t}{1+t}=\ln 2. \tag{8.3}$$

## Question 2

$$\lim_{\varepsilon\to 0^+} \varepsilon^4 \int_0^{\pi/2-\varepsilon} \tan^5 x\,\mathrm{d}x \tag{9.1}$$

**Solution** Denote the following function

$$f(\varepsilon)=\int_0^{\pi/2-\varepsilon} \tan^5 x\,\mathrm{d}x. \tag{9.2}$$

Its derivative can be analyzed from its **Laurent series**

$$f'(\varepsilon)=-\cot^5\varepsilon=-\left(\frac{1}{\varepsilon}-\frac{\varepsilon}{3}+O(\varepsilon)\right)^5=-\varepsilon^{-5}+O\left(\varepsilon^{-3}\right). \tag{9.3}$$

Therefore, we have

$$f(\varepsilon)=\frac{1}{4}\varepsilon^{-4}+O\left(\varepsilon^{-2}\right). \tag{9.4}$$

The original problem can be evaluated as

$$I=\lim_{\varepsilon\to 0^+}\varepsilon^4 f(\varepsilon)=\frac{1}{4}. \tag{9.5}$$

## Question 3

$$\int_0^1 \left\lfloor \sqrt{1+\frac{1}{x}} \right\rfloor \mathrm{d}x \tag{10.1}$$

**Solution** With the **change of variable**, we have

$$t = 1+\frac{1}{x} \in [2,+\infty], \qquad x = \frac{1}{t-1}, \qquad \mathrm{d}x = -\frac{1}{(t-1)^2}. \tag{10.2}$$

The integral becomes

$$\begin{aligned} I &= \int_2^{\infty} \frac{\lfloor \sqrt{t} \rfloor \, \mathrm{d}t}{(t-1)^2} = \int_2^4 \frac{\mathrm{d}t}{(t-1)^2} + \sum_{k=2}^{+\infty} \int_{k^2}^{(k+1)^2} \frac{k\,\mathrm{d}t}{(t-1)^2} \\ &= \frac{2}{3} + \sum_{k=2}^{+\infty} k \left[ \frac{1}{k^2-1} - \frac{1}{(k+1)^2-1} \right] \\ &= \frac{2}{3} + \sum_{k=2}^{+\infty} \left[ \frac{1}{(k-1)(k+1)} + \frac{1}{(k+1)(k+2)} \right]. \end{aligned} \tag{10.3}$$

Eventually, we obtain the result

$$I = \frac{2}{3} + \frac{1}{2}\left(1+\frac{1}{2}\right) + \frac{1}{3} = \frac{7}{4}. \tag{10.4}$$