MIT Integration Bee: 2023 Final

Question 1

$$\int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sin x + \cos x)^2}\, dx \tag{1.1}$$

Solution With the **change of variable** $t = \sqrt[3]{\tan x}$, we have

$$t = \sqrt[3]{\tan x}, \qquad 3t^2 dt = \sec^2 x\, dx. \tag{1.2}$$

Therefore, the integral becomes

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sin x + \cos x)^2}\, dx = \int_0^{\infty} \frac{3t^3\, dt}{(\sin x + \cos x)^2 \sec^2 x} \\ &= \int_0^{\infty} \frac{3t^3\, dt}{(t^3+1)^2} = -\frac{t}{t^3+1}\bigg|_0^{\infty} + \int_0^{\infty} \frac{dt}{t^3+1} \\ &= \int_0^{\infty} \frac{dt}{t^3+1} = \frac{2\pi}{3\sqrt{3}}. \end{aligned} \tag{1.3}$$

Note The following integral can be calculated using the **residue theorem**.

$$I_n = \int_0^{\infty} \frac{dx}{x^n+1}, \qquad \text{for } n \geq 2. \tag{1.4}$$

Based on the contour in Fig. 1, we have

$$\left(1 - e^{i2\pi/n}\right) I_n = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{z^n+1}, z_1\right), \qquad I_n = \frac{\pi}{n \sin(\pi/n)}. \tag{1.5}$$

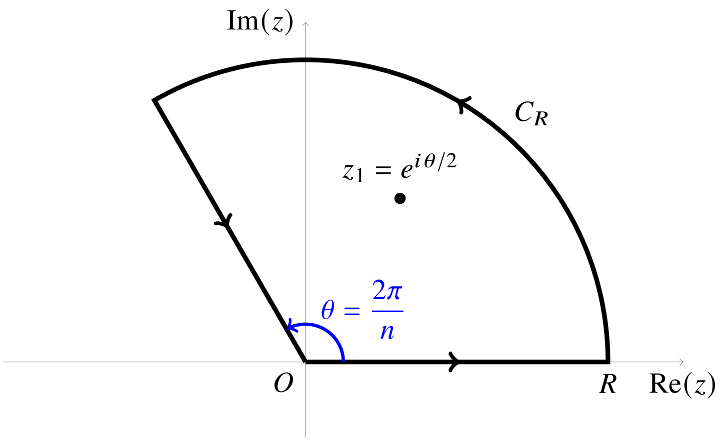

Fig. 1 Sector-shaped contour in the upper plane for $n = 3$. There is one simple pole z_1 within the contour.

Question 2

$$\int_0^{\pi} \left(\frac{\sin 2x \sin 3x \sin 5x \sin 30x}{\sin x \sin 6x \sin 10x \sin 15x} \right)^2 \, \mathrm{d}x \tag{2.1}$$

Solution Using double- and triple-angle formulae, we have

$$\begin{aligned} I &= \int_0^{\pi} \left(\frac{\cos x \cos 15x}{\cos 3x \cos 5x} \right)^2 \, \mathrm{d}x = \int_0^{\pi} \left(\frac{4\cos^2 5x - 3}{4\cos^2 x - 3} \right)^2 \, \mathrm{d}x \\ &= \int_0^{\pi} \left(\frac{2\cos 10x - 1}{2\cos 2x - 1} \right)^2 \, \mathrm{d}x = \frac{1}{2} \int_0^{2\pi} \left(\frac{2\cos 5x - 1}{2\cos x - 1} \right)^2 \, \mathrm{d}x \end{aligned} \tag{2.2}$$

We consider the following **Fourier series**

$$\frac{2\cos 5x - 1}{2\cos x - 1} = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x. \tag{2.3}$$

This leads to

$$\begin{aligned} 2\cos 5x = (-a_0 + a_1 + 1) + (2a_0 - a_1 + a_2)\cos x + (a_1 - a_2 + a_3)\cos 2x \\ + (a_2 - a_3 + a_4)\cos 3x + (a_3 - a_4)\cos 4x + a_4 \cos 5x. \end{aligned} \tag{2.4}$$

and matching the coefficients gives

$$\frac{2\cos 5x - 1}{2\cos x - 1} = -1 - 2\cos x + 2\cos 3x + 2\cos 4x. \tag{2.5}$$

Based on the orthogonality of the Fourier basis, the integral is calculated as

$$\begin{aligned} I &= \frac{1}{2} \int_0^{2\pi} (-1 - 2\cos x + 2\cos 3x + 2\cos 4x)^2 \, \mathrm{d}x \\ &= \frac{1}{2} \left(2\pi + 2^2 \cdot \pi + 2^2 \cdot \pi + 2^2 \cdot \pi \right) = 7\pi. \end{aligned} \tag{2.6}$$

Question 3

$$\int_{-1/2}^{1/2} \sqrt{x^2 + 1 + \sqrt{x^4 + x^2 + 1}}\, \mathrm{d}x \tag{3.1}$$

Solution We notice that

$$x^4 + x^2 + 1 = \left(x^2 + 1\right)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1). \tag{3.2}$$

Therefore, denote $u = x^2 + x + 1$ and $v = x^2 - x + 1$, we have

$$\begin{aligned} I &= \frac{1}{\sqrt{2}} \int_{-1/2}^{1/2} \sqrt{u + v + 2\sqrt{uv}}\, \mathrm{d}x = \frac{1}{\sqrt{2}} \int_{-1/2}^{1/2} \left(\sqrt{u} + \sqrt{v}\right)\, \mathrm{d}x \\ &= \sqrt{2} \int_{-1/2}^{1/2} \sqrt{x^2 + x + 1}\, \mathrm{d}x = \sqrt{2} \int_{0}^{1} \sqrt{x^2 + \frac{3}{4}}\, \mathrm{d}x. \end{aligned} \tag{3.3}$$

Based on the following indefinite integral

$$\int \sqrt{x^2 + a^2}\, \mathrm{d}x = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| + C, \tag{3.4}$$

we can see that the function evaluation at $x = 0$ becomes 0, and the result is

$$I = \frac{\sqrt{2}}{2} \left(\sqrt{\frac{7}{4}} + \frac{3}{4} \ln \left| \frac{2 + \sqrt{7}}{\sqrt{3}} \right| \right) = \frac{\sqrt{14}}{4} + \frac{3\sqrt{2}}{8} \ln \left(\frac{2 + \sqrt{7}}{\sqrt{3}} \right). \tag{3.5}$$

Question 4

$$\left\lfloor 10^{20} \int_2^{\infty} \frac{x^9}{x^{20} - 48x^{10} + 575} \, \mathrm{d}x \right\rfloor \tag{4.1}$$

Solution With $t = x^{10}$, we notice that

$$\int_2^{\infty} \frac{x^9}{x^{20} - 48x^{10} + 575} \, \mathrm{d}x = \frac{1}{10} \int_{2^{10}}^{\infty} \frac{\mathrm{d}t}{(t-24)^2 - 1} = \frac{1}{20} \ln \left(\frac{1001}{999} \right). \tag{4.2}$$

Based on the following **Taylor expansion**

$$\ln \left(\frac{a+x}{a-x} \right) = 2 \left(\frac{x}{a} \right) + \frac{2}{3} \left(\frac{x}{a} \right)^3 + \frac{2}{5} \left(\frac{x}{a} \right)^5 + \cdots , \tag{4.3}$$

we finally obtain

$$I = \left\lfloor \frac{10^{19}}{2} \ln \left(\frac{10^3 + 1}{10^3 - 1} \right) \right\rfloor = 10^{16} + \frac{10^{10} - 1}{3} + \frac{10^4}{5}. \tag{4.4}$$

Question 5

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{3^n} \right)^2 \mathrm{d}x \tag{5.1}$$

Solution For a real number $x \in (0, 1)$, its **binary representation** is (see 2024 Final: Question 5)

$$x = \sum_{k=1}^{+\infty} \frac{a_k}{2^k} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_k}{2^k} + \cdots, \qquad \text{with } a_k \in \{0, 1\}. \tag{5.2}$$

When $x \in (0, 1)$, we have i.i.d. random variables a_k following the two-point distribution with equal probability. With this representation, we have

$$\lfloor 2^n x \rfloor = 2^{n-1} a_1 + 2^{n-2} a_2 + \cdots + a_n, \tag{5.3}$$

and the summation can be computed as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{3^n} &= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{2^{n-m} a_m}{3^n} = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{2^{n-m} a_m}{3^n} \\ &= \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} \frac{2^l a_m}{3^{l+m}} = \sum_{m=1}^{\infty} \frac{a_m}{3^{m-1}}. \end{aligned} \tag{5.4}$$

The integral is now equivalent to an **expectation**, which is shown as follows

$$\begin{aligned} I &= \sum_{m=1}^{\infty} \frac{\mathbb{E}\left(a_m^2\right)}{9^{m-1}} + 2 \sum_{1 \le i < j} \frac{\mathbb{E}\left(a_i a_j\right)}{3^{i+j-2}} \\ &= \frac{9}{8} \mathbb{E}\left(a_m^2\right) + 2\, \mathbb{E}\left(a_i a_j\right) \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{3^{2i+l-2}} \\ &= \frac{9}{8} \mathbb{E}\left(a_m^2\right) + 2\, \mathbb{E}\left(a_i a_j\right) \cdot \frac{9}{8} \cdot \frac{1}{2} = \frac{9}{8} \left[\mathbb{E}\left(a_m^2\right) + \mathbb{E}\left(a_i a_j\right) \right]. \end{aligned} \tag{5.5}$$

Since a_k are i.i.d. and each follows the two-point distribution, we have

$$\mathbb{E}\left(a_m^2\right) = \frac{1}{2}, \qquad \mathbb{E}\left(a_i a_j\right) = \mathbb{E}\left(a_i\right) \mathbb{E}\left(a_j\right) = \frac{1}{4}. \tag{5.6}$$

Finally, we obtain

$$I = \frac{9}{8} \left[\mathbb{E}\left(a_m^2\right) + \mathbb{E}\left(a_i a_j\right) \right] = \frac{27}{32}. \tag{5.7}$$