MIT Integration Bee: 2022 Semifinal

Semifinal #1

Question 1

$$\int_0^{+\infty} \frac{x\left(e^{-x}+1\right)}{e^x-1}\,\mathrm{d}x \tag{1.1}$$

Solution The integral can be decomposed into two simpler integrals

$$I = -\int_0^{+\infty} xe^{-x}\,\mathrm{d}x + \int_0^{+\infty} \frac{2x}{e^x-1}\,\mathrm{d}x = -1 + 2\cdot\frac{\pi^2}{6} = \frac{\pi^2}{3} - 1. \tag{1.2}$$

Note The second integral is in fact related to the **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{x^{s-1}}{e^x-1}\,\mathrm{d}x. \tag{1.3}$$

Therefore, we have

$$\int_0^{+\infty} \frac{x}{e^x-1}\,\mathrm{d}x = \Gamma(2)\zeta(2) = \frac{\pi^2}{6}. \tag{1.4}$$

Other integral representations of $\zeta(2)$, also related to the **Basel problem**, include

$$\zeta(2) = -\int_0^1 \frac{\ln x}{1-x}\,\mathrm{d}x = \int_0^1 \frac{\ln^2 x}{(1+x)^2}\,\mathrm{d}x. \tag{1.5}$$

Question 2

$$\int_0^{+\infty} x^5 e^{-x} \sin x \, \mathrm{d}x \tag{2.1}$$

Solution Denote the following integrals

$$I_n = \int_0^{+\infty} x^n e^{-x} \sin x \, \mathrm{d}x, \qquad J_n = \int_0^{+\infty} x^n e^{-x} \cos x \, \mathrm{d}x, \qquad \text{for } n \in \mathbb{N}. \tag{2.2}$$

When $n \geq 1$, the **reduction formula** can be obtained as

$$I_n = nJ_{n-1} - J_n, \qquad J_n = -nI_{n-1} + I_n. \tag{2.3}$$

We can also write them as

$$I_n = \frac{n}{2}\left(I_{n-1} + J_{n-1}\right), \qquad J_n = \frac{n}{2}\left(-I_{n-1} + J_{n-1}\right). \tag{2.4}$$

Since we have

$$I_0 = \int_0^{+\infty} e^{-x} \sin x \, \mathrm{d}x = 1 - J_0, \qquad J_0 = I_0, \qquad \Longrightarrow \qquad I_0 = J_0 = \frac{1}{2}, \tag{2.5}$$

we can recursively obtain

$$I_1 = \frac{1}{2}, \quad J_1 = 0, \quad I_2 = \frac{1}{2}, \quad J_2 = -\frac{1}{2}, \quad I_3 = 0, \quad J_3 = -\frac{3}{2}, \quad I_4 = -3, \quad J_4 = -3. \tag{2.6}$$

Eventually, we have

$$I_5 = \frac{5}{2}\left(I_4 + J_4\right) = -15. \tag{2.7}$$

Question 3

$$\int_{1/2}^{2} \ln\left(\frac{\ln\left(x+\frac{1}{x}\right)}{\ln\left(x^2-x+\frac{17}{4}\right)}\right) \mathrm{d}x \tag{3.1}$$

Solution The integral can be decomposed into several parts

$$\begin{aligned} I &= \int_{1/2}^{2} \ln\frac{1}{2}\,\mathrm{d}x + \int_{1/2}^{2} \ln\left(\ln\left[\left(x-\frac{1}{x}\right)^2+4\right]\right)\mathrm{d}x - \int_{1/2}^{2} \ln\left(\ln\left[\left(x-\frac{1}{2}\right)^2+4\right]\right)\mathrm{d}x \\ &= -\frac{3}{2}\ln 2 + J - K. \end{aligned} \tag{3.2}$$

Now, we show $J = K$. For integral J, a change of variable $x = e^t$ gives

$$J = \int_{-\ln 2}^{\ln 2} \ln\ln\left(4\cosh^2 t\right) e^t\,\mathrm{d}t = 2\int_0^{\ln 2} \ln\ln\left(4\cosh^2 t\right)\,\cosh t\,\mathrm{d}t. \tag{3.3}$$

For integral K, we consider the following **change of variable**

$$x - \frac{1}{2} = 2\sinh u, \qquad u(x) = \sinh^{-1}\left(\frac{x}{2}-\frac{1}{4}\right), \qquad u\left(\frac{1}{2}\right) = 0, \qquad u(2) = \ln 2. \tag{3.4}$$

Note that the inverse hyperbolic function is evaluated as

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right). \tag{3.5}$$

Therefore, we obtain the same result

$$K = 2\int_0^{\ln 2} \ln\ln\left(4\cosh^2 t\right)\,\cosh t\,\mathrm{d}t. \tag{3.6}$$

Eventually, we have

$$I = -\frac{3}{2}\ln 2. \tag{3.7}$$

Question 4

$$\int_{2}^{5/2} \frac{\left(x^3 - 3x\right)^3 - 3\left(x^3 - 3x\right)}{\sqrt{x^2 - 4}}\, \mathrm{d}x \tag{4.1}$$

Solution With a **change of variable** $t = \sqrt{x^2 - 4}$, we have

$$x = \sqrt{t^2 + 4}, \qquad \mathrm{d}x = \frac{t\, \mathrm{d}t}{\sqrt{t^2 + 4}} = \frac{t}{x}\, \mathrm{d}t. \tag{4.2}$$

Now, the integral becomes

$$\begin{aligned} I &= \int_{2}^{5/2} \frac{x^3\left(x^2 - 3\right)^3 - 3x\left(x^2 - 3\right)}{t}\, \mathrm{d}x \\ &= \int_{0}^{3/2} \left[x^2\left(x^2 - 3\right)^3 - 3\left(x^2 - 3\right)\right] \mathrm{d}t \\ &= \int_{0}^{3/2} \left[\left(t^2 + 4\right)\left(t^2 + 1\right)^3 - 3\left(t^2 + 1\right)\right] \mathrm{d}t \\ &= \int_{0}^{3/2} \left(t^8 + 7t^6 + 15t^4 + 10t^2 + 1\right) \mathrm{d}t \end{aligned} \tag{4.3}$$

We can evaluate the integral and obtain the following result

$$\begin{aligned} I &= \left[\frac{t^9}{9} + t^7 + 3t^5 + \frac{10}{3}t^3 + t\right]_{t=0}^{t=3/2} \\ &= \frac{1}{9}\left(\frac{3^9}{2^9} + \frac{3^9}{2^7} + \frac{3^8}{2^5} + \frac{5 \times 3^4}{2^2} + \frac{3^3}{2}\right) \\ &= \frac{1}{9 \times 2^9}\left(3^9 + 2^2 \times 3^9 + 2^4 \times 3^8\right) + \frac{1}{9 \times 2^2}\left(5 \times 3^4 + 2 \times 3^3\right) \\ &= \frac{262143}{9 \times 2^9} = \frac{2^9 - 2^{-9}}{9}. \end{aligned} \tag{4.4}$$

Semifinal #2

Question 1

$$\int_{-1}^{1} \left|\left|\left||x| - \frac{2}{3}\right| - \frac{2}{3^2}\right| - \frac{2}{3^3}\right| - \cdots \right| \mathrm{d}x \tag{5.1}$$

Solution Denote the integrand as $f(x)$, and its graph is a **fractal** shape. Based on symmetry, we have

$$I = 2\int_{0}^{1} \left|\left|\left|x - \frac{2}{3}\right| - \frac{2}{3^2}\right| - \frac{2}{3^3}\right| - \cdots \right| \mathrm{d}x = 2\int_0^1 f(x)\,\mathrm{d}x. \tag{5.2}$$

Now denote the following infinite sequence and its sum

$$a_n = \frac{2}{3^n}, \qquad S = \lim_{n\to\infty} S_n = 1. \tag{5.3}$$

We can obtain some special values of the function $f(x)$ as follows

$$f(0) = a_1 - (S - a_1) = \frac{1}{3}, \qquad f\left(\frac{1}{3}\right) = 0, \qquad f(1) = 0. \tag{5.4}$$

The integration within the sub-interval $[0, 1/3]$ is trivial. For the rest of the interval, note that

$$f\left(x + \frac{2}{3}\right) = f\left(\frac{2}{3} - x\right) = \left|\left|x - \frac{2}{3^2}\right| - \frac{2}{3^3}\right| - \cdots \right| = \frac{1}{3} f(3x), \qquad \text{for } x \in \left[0, \frac{1}{3}\right]. \tag{5.5}$$

Therefore, we have

$$\begin{aligned} I &= 2\int_0^{1/3} f(x)\,\mathrm{d}x + 4\int_0^{1/3} \frac{1}{3} f(3x)\,\mathrm{d}x \\ &= \frac{1}{9} + \frac{4}{9}\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{9} + \frac{2}{9} I. \end{aligned} \tag{5.6}$$

The final result is thus obtained as

$$I = \frac{1}{9} + \frac{2}{9} I, \qquad I = \frac{1}{7}. \tag{5.7}$$

Question 2

$$\int \frac{1}{\left(x^2+1\right)^3}\,\mathrm{d}x \tag{6.1}$$

Solution Based on the following **change of variable**

$$x = \sinh t, \qquad x^2+1 = \cosh^2 t, \qquad \mathrm{d}x = \cosh t\,\mathrm{d}t, \tag{6.2}$$

the integral now becomes

$$I_5 = \int \operatorname{sech}^5 t\,\mathrm{d}t. \tag{6.3}$$

For $n \geq 1$, the **reduction formula** can be obtained as

$$I_n = \frac{1}{n-1}\operatorname{sech}^{n-2} t\,\tanh t + \frac{n-2}{n-1} I_{n-2}, \qquad I_1 = \arctan(\sinh t) + C. \tag{6.4}$$

Therefore, we have

$$I_3 = \frac{1}{2}\operatorname{sech} t\,\tanh t + \frac{1}{2}\arctan(\sinh t) + C, \tag{6.5}$$

$$I_5 = \frac{1}{4}\operatorname{sech}^3 t\,\tanh t + \frac{3}{8}\operatorname{sech} t\,\tanh t + \frac{3}{8}\arctan(\sinh t) + C. \tag{6.6}$$

Using the following relationships

$$\operatorname{sech}^2 t = \frac{1}{x^2+1}, \qquad \operatorname{sech} t\,\tanh t = \frac{x}{x^2+1}, \tag{6.7}$$

the original integral is calculated as

$$I = \frac{3}{8}\arctan x + \frac{3x^3+5x}{8\left(x^2+1\right)^2} + C. \tag{6.8}$$

Question 3

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2 - 2x\cot x + \csc^2 x} \tag{7.1}$$

Solution Based on the **Glasser's master theorem** (see 2023 Quarterfinal #3: Question 2 and 2024 Semifinal #1: Tiebreaker 2), the integral is equivalent to

$$I = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x - \cot x)^2 + 1} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2 + 1} = \pi. \tag{7.2}$$

Question 4

$$\int_0^{\pi/6} \ln\left(\sqrt{3} + \tan x\right) \mathrm{d}x \tag{8.1}$$

Solution

$$\begin{aligned} I &= \int_0^{\pi/6} \ln\left[\frac{2\cos(x - \pi/6)}{\cos x}\right] \mathrm{d}x \\ &= \frac{\pi}{6}\ln 2 + \int_0^{\pi/6} \ln\left[\cos\left(x - \frac{\pi}{6}\right)\right] \mathrm{d}x - \int_0^{\pi/6} \ln\cos x \,\mathrm{d}x. \end{aligned} \tag{8.2}$$

For the second term, we can show that

$$\int_0^{\pi/6} \ln\left[\cos\left(x - \frac{\pi}{6}\right)\right] \mathrm{d}x = \int_{-\pi/6}^{0} \ln\cos x \,\mathrm{d}x = \int_0^{\pi/6} \ln\cos x \,\mathrm{d}x. \tag{8.3}$$

Finally, we obtain the result of the integral

$$I = \frac{\pi}{6}\ln 2. \tag{8.4}$$