MIT Integration Bee: 2022 Regular Season

Question 1

$$\int_0^{100} \left\lceil \sqrt{x} \right\rceil \mathrm{d}x \tag{1.1}$$

Solution

$$\begin{aligned} I &= \int_0^1 1\,\mathrm{d}x + \int_1^4 2\,\mathrm{d}x + \cdots + \int_{81}^{100} 10\,\mathrm{d}x = \sum_{k=1}^{10} k\left[k^2 - (k-1)^2\right] \\ &= 2\sum_{k=1}^{10} k^2 - \sum_{k=1}^{10} k = 715. \end{aligned} \tag{1.2}$$

Question 2

$$\int \frac{\ln(1+x)}{x^2}\,\mathrm{d}x \tag{2.1}$$

Solution

$$I = -\frac{\ln(1+x)}{x} + \int \frac{\mathrm{d}x}{x(x+1)} = -\frac{\ln(1+x)}{x} + \ln\left(\frac{x}{x+1}\right) + C. \tag{2.2}$$

Question 3

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}+1} \cos(\arcsin(\arccos(\sin x)))\,\mathrm{d}x \tag{3.1}$$

Solution

$$I = \int_{-1}^{1} \cos(\arcsin t)\,\mathrm{d}t = \int_{-1}^{1} \sqrt{1-t^2}\,\mathrm{d}t = \frac{\pi}{2}. \tag{3.2}$$

Question 4

$$\int_{-2}^{2} |(x-2)(x-1)\,x\,(x+1)(x+2)|\;\mathrm{d}x \tag{4.1}$$

Solution

$$I = 2\left[\int_0^1 \left(x^5 - 5x^3 + 4x\right)\mathrm{d}x - \int_1^2 \left(x^5 - 5x^3 + 4x\right)\mathrm{d}x\right] = \frac{19}{3}. \tag{4.2}$$

Question 5

$$\int \left[2020\sin^{2019}(x)\cos^{2019}(x) - 8084\sin^{2021}(x)\cos^{2021}(x)\right]\mathrm{d}x \tag{5.1}$$

Solution Note that the integrand can be written as

$$\frac{1}{2^{2019}}\sin^{2019}(2x)\left[2020 - 2021\sin^2(2x)\right]. \tag{5.2}$$

Since the following integral can be evaluated as

$$\begin{aligned} I_{2021} &= \int \sin^{2021}(2x)\,\mathrm{d}x \\ &= -\frac{1}{2}\sin^{2020}(2x)\cos(2x) + 2020\int \sin^{2019}(2x)\cos^2(2x)\,\mathrm{d}x \\ &= -\frac{1}{2}\sin^{2020}(2x)\cos(2x) + 2020\left(I_{2019} - I_{2021}\right), \end{aligned} \tag{5.3}$$

for the original problem, we thus have

$$\begin{aligned} I &= \frac{1}{2^{2019}}\left(2020\,I_{2019} - 2021\,I_{2021}\right) \\ &= \frac{1}{2^{2020}}\sin^{2020}(2x)\cos(2x) \\ &= \sin^{2020}(x)\cos^{2020}(x)\cos(2x) \\ &= \sin^{2020}(x)\cos^{2022}(x) - \sin^{2022}(x)\cos^{2020}(x). \end{aligned} \tag{5.4}$$

Question 6

$$\int \frac{3x^3 + 2x^2 + 1}{\sqrt[3]{x^3 + 1}} \, \mathrm{d}x \tag{6.1}$$

Solution Denote the following function

$$f(x) = \left(x^3 + 1\right)^{\frac{2}{3}} = \frac{x^3 + 1}{\sqrt[3]{x^3 + 1}}, \qquad f'(x) = \frac{2x^2}{\sqrt[3]{x^3 + 1}}. \tag{6.2}$$

Therefore, the integrand can be written as

$$\frac{3x^3 + 2x^2 + 1}{\sqrt[3]{x^3 + 1}} = (x + 1) f'(x) + f(x), \tag{6.3}$$

which directly gives the result of the original problem

$$I = (x + 1) f(x) + C = (x + 1) \left(x^3 + 1\right)^{\frac{2}{3}} + C. \tag{6.4}$$

Question 7

$$\int \frac{\mathrm{d}x}{\sin^4 x \cos^4 x} \tag{7.1}$$

Solution Denote the following integral, and we can obtain the **reduction formula**

$$I_n = \int \csc^n (2x) \, \mathrm{d}x = I_{n-2} - \frac{1}{2(n-1)} \cot^{n-1} (2x), \qquad I_2 = -\frac{1}{2} \cot (2x). \tag{7.2}$$

Hence, the original problem is solved as

$$I = 16 \, I_4 = -8 \cot (2x) - \frac{8}{3} \cot^3 (2x) + C. \tag{7.3}$$

Question 8

$$\int \frac{x + \sin x}{1 + \cos x} \, \mathrm{d}x \tag{8.1}$$

Solution

$$\begin{aligned} I &= \int \frac{x}{1 + \cos x} \, \mathrm{d}x + \int \tan \frac{x}{2} \, \mathrm{d}x = \int \frac{x}{2} \sec^2 \frac{x}{2} \, \mathrm{d}x + \int \tan \frac{x}{2} \, \mathrm{d}x \\ &= x \tan \frac{x}{2} - \int \tan \frac{x}{2} \, \mathrm{d}x + \int \tan \frac{x}{2} \, \mathrm{d}x = x \tan \frac{x}{2} + C. \end{aligned} \tag{8.2}$$

Question 9

$$\int \sinh^3 x \cosh^2 x \, \mathrm{d}x \tag{9.1}$$

Solution

$$I = \int \left(\cosh^2 x - 1\right) \cosh^2 x \, \mathrm{d}\left(\cosh x\right) = \frac{1}{5}\cosh^5 x - \frac{1}{3}\cosh^3 x + C. \tag{9.2}$$

Question 10

$$\int 4^x \cdot 3^{2^x} \, \mathrm{d}x \tag{10.1}$$

Solution Based on the following derivative pair

$$f(x) = 3^{2^x}, \qquad f'(x) = (\ln 2)(\ln 3)\; 2^x \cdot 3^{2^x}. \tag{10.2}$$

the integral can be evaluated as

$$\begin{aligned} I &= \frac{1}{(\ln 2)(\ln 3)} \int 2^x \, \mathrm{d}\left(3^{2^x}\right) = \frac{2^x \cdot 3^{2^x}}{(\ln 2)(\ln 3)} - \frac{1}{\ln 3}\int 2^x \cdot 3^{2^x} \, \mathrm{d}x \\ &= \frac{2^x \cdot 3^{2^x}}{(\ln 2)(\ln 3)} - \frac{3^{2^x}}{(\ln 2)(\ln 3)^2} + C. \end{aligned} \tag{10.3}$$

Question 11

$$\int \frac{\cos x - \sin x}{2 + \sin 2x} \, \mathrm{d}x \tag{11.1}$$

Solution

$$I = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2 + 1} \, \mathrm{d}x = \arctan\left(\sin x + \cos x\right) + C. \tag{11.2}$$

Question 12

$$\int \frac{\sec^2(1 + \ln x) - \tan(1 + \ln x)}{x^2} \, \mathrm{d}x \tag{12.1}$$

Solution

$$I = \int \frac{1}{x} \, \mathrm{d}\left[\tan(1 + \ln x)\right] - \int \frac{\tan(1 + \ln x)}{x^2} \, \mathrm{d}x = \frac{\tan(1 + \ln x)}{x} + C. \tag{12.2}$$

Question 13

$$\int_0^1 \sqrt{\frac{1}{x}\ln\left(\frac{1}{x}\right)}\,\mathrm{d}x \tag{13.1}$$

Solution Based on the following **change of variable**

$$t = -\frac{1}{2}\ln x, \qquad x = e^{-2t}, \qquad \mathrm{d}x = -2e^{-2t}\,\mathrm{d}t, \tag{13.2}$$

the integral becomes

$$I = 2\sqrt{2}\int_0^{+\infty} \sqrt{t}e^{-t}\,\mathrm{d}t = 2\sqrt{2}\,\Gamma\left(\frac{3}{2}\right) = \sqrt{2\pi}. \tag{13.3}$$

Question 14

$$\sum_{n=2}^{+\infty}\int_0^{+\infty} \frac{(x-1)\,x^n}{1+x^n+x^{n+1}+x^{2n+1}}\,\mathrm{d}x \tag{14.1}$$

Solution

$$\begin{aligned} I &= \sum_{n=2}^{+\infty}\int_0^{+\infty}\left(\frac{1}{x^n+1} - \frac{1}{x^{n+1}+1}\right)\mathrm{d}x \\ &= \int_0^{+\infty}\frac{\mathrm{d}x}{x^2+1} - \lim_{n\to\infty}\int_0^{+\infty}\frac{\mathrm{d}x}{x^n+1} = \frac{\pi}{2} - 1. \end{aligned} \tag{14.2}$$

Note that for the second term, the contribution from the interval $[0, 1]$ needs to be considered when evaluating the limit.

Question 15

$$\int_0^{2\pi} (1-\cos x)^5\cos(5x)\,\mathrm{d}x \tag{15.1}$$

Solution We only need to find the constant term for the integrand. This is done by

$$\begin{aligned} (1-\cos x)^5\cos(5x) &= \frac{1}{2^6}\left(2 - e^{ix} - e^{-ix}\right)^5\left(e^{i5x} + e^{-i5x}\right) \\ &= \frac{1}{2^6}\left(-2 + f\left(e^{inx}\right)\right). \end{aligned} \tag{15.2}$$

Therefore, the integral is evaluated as

$$I = -\frac{1}{2^5}\cdot 2\pi = -\frac{\pi}{16}. \tag{15.3}$$

Question 16

$$\int_0^{10} \lceil x \rceil \left(\max_{k\in\mathbb{N}} \frac{x^k}{k!} \right) \mathrm{d}x \tag{16.1}$$

Solution

$$\begin{aligned} I &= \sum_{n=0}^{9} \int_n^{n+1} (n+1) \cdot \frac{x^n}{n!} \,\mathrm{d}x = \sum_{n=0}^{9} \frac{(n+1)^{n+1} - n^{n+1}}{n!} \\ &= \sum_{n=0}^{9} \frac{(n+1)^{n+1}}{n!} - \sum_{n=0}^{8} \frac{(n+1)^{n+1}}{n!} = \frac{10^{10}}{9!}. \end{aligned} \tag{16.2}$$

Question 17

$$\int \frac{4\sin x + 3\cos x}{3\sin x + 4\cos x} \,\mathrm{d}x \tag{17.1}$$

Solution Denote the following functions

$$f(x) = 3\sin x + 4\cos x, \qquad f'(x) = -4\sin x + 3\cos x, \qquad g(x) = 4\sin x + 3\cos x. \tag{17.2}$$

Now consider

$$g(x) = Af(x) + Bf'(x), \tag{17.3}$$

and we can solve the coefficients as

$$A = \frac{24}{25}, \qquad B = -\frac{7}{25}, \qquad I = \int \frac{Af(x) + Bf'(x)}{f(x)} \,\mathrm{d}x. \tag{17.4}$$

The integral can thus be solved as

$$I = \frac{24x - 7\ln(3\sin x + 4\cos x)}{25} + C. \tag{17.5}$$

Question 18

$$\int_{-1}^{1} \left[\sqrt{4-(1+|x|)^2} - \left(\sqrt{3} - \sqrt{4-x^2} \right) \right] \mathrm{d}x \tag{18.1}$$

Solution Consider the graph of the integrand. It is symmetrical with respect to y-axis, and the integral can be decomposed into a combination of circular sectors and triangles.

$$I = 2 \times \left(\frac{1}{2} \cdot 2^2 \cdot \frac{\pi}{6} - \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot 2^2 \cdot \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = 2\pi - 2\sqrt{3}. \tag{18.2}$$

Question 19

$$\int x^2 \sin(\ln x) \, \mathrm{d}x \tag{19.1}$$

Solution

$$\begin{aligned} I &= \int x^2 \sin(\ln x) \, \mathrm{d}x = \frac{x^3}{3} \sin(\ln x) - \int \frac{x^2}{3} \cos(\ln x) \, \mathrm{d}x \\ &= \frac{x^3}{3} \sin(\ln x) - \frac{x^3}{9} \cos(\ln x) - \frac{I}{9}. \end{aligned} \tag{19.2}$$

Therefore, the integral is solved as

$$I = \frac{x^3}{10} \left[3\sin(\ln x) - \cos(\ln x) \right] + C. \tag{19.3}$$

Question 20

$$\int_{0}^{+\infty} \left(36x^5 - 12x^6 + x^7 \right) e^{-x} \, \mathrm{d}x \tag{20.1}$$

Solution Based on the following result (see 2024 Regular Season: Question 20)

$$I_n = \int_{0}^{\infty} a_n x^n e^{-x} \, \mathrm{d}x = n! \cdot a_n, \tag{20.2}$$

we have

$$I = 36 \cdot 5! - 12 \cdot 6! + 7! = 6! = 720. \tag{20.3}$$