MIT Integration Bee: 2022 Quarterfinal

Quarterfinal #1

Question 1

$$\int_1^{2022} \frac{\{x\}}{x}\,\mathrm{d}x \tag{1.1}$$

Solution

$$\begin{aligned} I &= \sum_{k=1}^{2021} \int_k^{k+1} \frac{x-k}{x}\,\mathrm{d}x = 2021 - \sum_{k=1}^{2021} \int_k^{k+1} \frac{k}{x}\,\mathrm{d}x \\ &= 2021 - \sum_{k=1}^{2021} k\left[\ln(k+1) - \ln k\right] \\ &= 2021 - 2021\ln 2022 + \ln 2021! = 2021 - \ln\left(\frac{2022^{2021}}{2021!}\right). \end{aligned} \tag{1.2}$$

Question 2

$$\lim_{n\to\infty} n \int_0^{\pi/4} \tan^n x\,\mathrm{d}x \tag{2.1}$$

Solution Denote the following definite integral

$$I_n = \int_0^{\pi/4} \tan^n x\,\mathrm{d}x, \qquad I_0 = \frac{\pi}{4}, \qquad I_1 = \frac{1}{2}\ln 2. \tag{2.2}$$

We can obtain the **reduction formula** as

$$I_n = \int_0^{\pi/4} \left(\sec^2 x - 1\right) \tan^{n-2} x\,\mathrm{d}x = \frac{1}{n-1} - I_{n-2}. \tag{2.3}$$

Based on this result, we have

$$\lim_{n\to\infty} n\,I_n = 1 - \lim_{n\to\infty} n\,I_n, \qquad \lim_{n\to\infty} n \int_0^{\pi/4} \tan^n x\,\mathrm{d}x = \lim_{n\to\infty} n\,I_n = \frac{1}{2}. \tag{2.4}$$

Question 3

$$\int_0^{+\infty} \frac{x^{1010}}{(1+x)^{2022}}\,\mathrm{d}x \tag{3.1}$$

Solution Using the following identity of the **Beta function**

$$\mathrm{B}(m,n) = \int_0^{+\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{d}x, \tag{3.2}$$

we have

$$I = \mathrm{B}(1011, 1011) = \frac{\Gamma^2(1011)}{\Gamma(2022)} = \frac{(1010!)^2}{2021!}. \tag{3.3}$$

Note The **Beta function** is related to many types of integrals. Some examples are shown below.

$$\mathrm{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,\mathrm{d}x \tag{3.4}$$

$$= \int_0^{+\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{d}x \tag{3.5}$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,\mathrm{d}\theta \tag{3.6}$$

$$= \alpha\int_0^1 x^{\alpha m-1}(1-x^{\alpha})^{n-1}\,\mathrm{d}x. \tag{3.7}$$

Quarterfinal #2

Question 1

$$\int \arcsin x \arccos x \, \mathrm{d}x \tag{4.1}$$

Solution

$$\begin{aligned} I &= x \arcsin x \arccos x - \int \frac{x}{\sqrt{1-x^2}} (\arccos x - \arcsin x) \, \mathrm{d}x \\ &= x \arcsin x \arccos x + \sqrt{1-x^2} (\arccos x - \arcsin x) + 2x + C. \end{aligned} \tag{4.2}$$

Question 2

$$\max_{\{x_i\}=\{1,2,3,4,5,6,7\}} \int_{\int_{x_1}^{x_2} x_3 \, \mathrm{d}x}^{\int_{x_4}^{x_5} x_6 \, \mathrm{d}x} x_7 \, \mathrm{d}x \tag{5.1}$$

Solution

$$I = \max_{\{x_i\}=\{1,2,3,4,5,6,7\}} [x_6 (x_5 - x_4) - x_3 (x_2 - x_1)] \, x_7. \tag{5.2}$$

When $(x_i) = (4, 2, 3, 1, 6, 5, 7)$, we obtain the maximal value

$$I = [5 \times (6-1) - 3 \times (2-4)] \times 7 = 31 \times 7 = 217. \tag{5.3}$$

Question 3

$$\lim_{n\to\infty} \sqrt[n]{\int_0^2 \left(1 + 6x - 7x^2 + 4x^3 - x^4\right)^n \, \mathrm{d}x} \tag{6.1}$$

Solution Note that

$$1 + 6x - 7x^2 + 4x^3 - x^4 = -(x-1)^4 - (x-1)^2 + 3 \tag{6.2}$$

With the **change of variable** $t = x - 1$, we have

$$3^n \le I_n = \int_{-1}^{1} \left(3 - t^2 - t^4\right)^n \, \mathrm{d}t \le 2 \cdot 3^n. \tag{6.3}$$

Taking the limit, we thus obtain

$$\lim_{n\to\infty} \sqrt[n]{I_n} = 3. \tag{6.4}$$

Quarterfinal #3

Question 1

$$\int_0^1 \frac{x^4}{\sqrt{1-x}} \mathrm{d}x \tag{7.1}$$

Solution

$$I = \mathrm{B}\left(5, \frac{1}{2}\right) = \frac{4! \times \sqrt{\pi}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} = \frac{256}{315}. \tag{7.2}$$

Question 2

$$\int_0^{+\infty} \left\lceil \frac{1}{\lceil x \rceil - x} \right\rceil^{-\lceil x \rceil} \mathrm{d}x \tag{8.1}$$

Solution

$$\begin{aligned} I &= \sum_{k=1}^{\infty} \int_0^1 \left\lceil \frac{1}{1-x} \right\rceil^{-k} \mathrm{d}x = \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) \cdot \frac{1}{n^k} \\ &= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) \cdot \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \frac{\pi^2}{6} - 1. \end{aligned} \tag{8.2}$$

Question 3

$$\lim_{n\to\infty} n \int_0^{+\infty} \sin\left(\frac{1}{x^n}\right) \mathrm{d}x \tag{9.1}$$

Solution With the following **change of variable**

$$t = \frac{1}{x^n}, \qquad x = t^{-\frac{1}{n}}, \qquad \mathrm{d}x = -\frac{1}{n}\, t^{-1-\frac{1}{n}}\, \mathrm{d}t, \tag{9.2}$$

we have

$$\lim_{n\to\infty} n\, I_n = \lim_{n\to\infty} \int_0^{+\infty} t^{-1-\frac{1}{n}} \sin t \,\mathrm{d}t = \int_0^{+\infty} \frac{\sin t}{t} \mathrm{d}t = \frac{\pi}{2}. \tag{9.3}$$

Quarterfinal #4

Question 1

$$\int_0^1 \frac{-x+\sqrt{4-3x^2}}{2}\,\mathrm{d}x \tag{10.1}$$

Solution With the **change of variable** $t=\sqrt{3}x$, we have

$$\int_0^1 \sqrt{4-3x^2}\,\mathrm{d}x = \frac{\sqrt{3}}{3}\int_0^{\sqrt{3}} \sqrt{4-t^2}\,\mathrm{d}t = \frac{2\sqrt{3}}{9}\pi + \frac{1}{2}. \tag{10.2}$$

The integral can thus be computed as

$$I = \frac{\sqrt{3}}{9}\pi + \frac{1}{4} - \frac{1}{4} = \frac{\sqrt{3}}{9}\pi. \tag{10.3}$$

Question 2

$$\lim_{n\to\infty} \sqrt{n}\int_{-1/2}^{1/2} \left(1-3x^2+x^4\right)^n \mathrm{d}x \tag{11.1}$$

Solution With a **change of variable** $t=x\sqrt{n}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n}\, I_n &= \lim_{n\to\infty} \int_{-\sqrt{n}/2}^{\sqrt{n}/2} \exp\left[n\ln\left(1-\frac{3t^2}{n}+\frac{t^4}{n^2}\right)\right]\mathrm{d}t \\
&= \lim_{n\to\infty} \int_{-\sqrt{n}/2}^{\sqrt{n}/2} \exp\left[-n\cdot\left(\frac{3t^2}{n}+\frac{7t^4}{2n^2}+\cdots\right)\right]\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} e^{-3t^2}\,\mathrm{d}t = \sqrt{\frac{\pi}{3}}.
\end{aligned} \tag{11.2}$$

Question 3

$$\int_{1/2022}^{2022} \frac{1+x^2}{x^2+x^{2022}}\,\mathrm{d}x \tag{12.1}$$

Solution With a **change of variable** $t=x^{-1}$, we have

$$I = \int_{1/2022}^{2022} \frac{1+x^2}{x^2+x^{2022}}\,\mathrm{d}x = \int_{1/2022}^{2022} \frac{t^{2020}\left(1+t^2\right)}{t^2+t^{2022}}\,\mathrm{d}t. \tag{12.2}$$

Therefore, the integral can be obtained as

$$I = \frac{1}{2}\int_{1/2022}^{2022} \frac{\left(1+t^{2020}\right)\left(1+t^2\right)}{t^2+t^{2022}}\,\mathrm{d}t = \frac{1}{2}\int_{1/2022}^{2022} \frac{1+t^2}{t^2}\,\mathrm{d}t = 2022 - \frac{1}{2022}. \tag{12.3}$$