MIT Integration Bee: 2022 Final

Question 1

$$\int \sqrt{[\sin(20x) + 3\sin(21x) + \sin(22x)]^2 + [\cos(20x) + 3\cos(21x) + \cos(22x)]^2}\,\mathrm{d}x \tag{1.1}$$

Solution

$$I = \int \sqrt{11 + 6\cos x + 6\cos x + 2\cos 2x}\,\mathrm{d}x = \int (2\cos x + 3)\,\mathrm{d}x = 2\sin x + 3x + C. \tag{1.2}$$

Question 2

$$\int_0^{+\infty} \frac{e^{-2x}\sin(3x)}{x}\,\mathrm{d}x \tag{2.1}$$

Solution Consider the general integral

$$I(\alpha,\beta) = \int_0^{+\infty} \frac{e^{-\alpha x}\sin(\beta x)}{x}\,\mathrm{d}x, \qquad I(0,\beta) = \frac{\pi}{2}. \tag{2.2}$$

Using the **Feynman's trick**, we have

$$\frac{\partial I}{\partial \alpha} = -F(\alpha,\beta) = -\int_0^{+\infty} e^{-\alpha x}\sin(\beta x)\,\mathrm{d}x. \tag{2.3}$$

The integral $F(\alpha,\beta)$ can be evaluated using **integration by parts** twice, which gives

$$\begin{aligned} F(\alpha,\beta) &= -\frac{1}{\alpha}e^{-\alpha x}\sin(\beta x)\Big|_0^{+\infty} + \frac{\beta}{\alpha}\int_0^{+\infty} e^{-\alpha x}\cos(\beta x)\,\mathrm{d}x \\ &= -\frac{\beta}{\alpha^2}e^{-\alpha x}\cos(\beta x)\Big|_0^{+\infty} - \frac{\beta^2}{\alpha^2}\int_0^{+\infty} e^{-\alpha x}\sin(\beta x)\,\mathrm{d}x = \frac{\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2}F(\alpha,\beta). \end{aligned} \tag{2.4}$$

Hence, we have

$$F(\alpha,\beta) = \frac{\beta}{\alpha^2+\beta^2}, \qquad I(\alpha,\beta) = \frac{\pi}{2} - \arctan\left(\frac{\alpha}{\beta}\right) = \arctan\left(\frac{\beta}{\alpha}\right). \tag{2.5}$$

The original integral becomes

$$I = I(2,3) = \arctan\left(\frac{3}{2}\right). \tag{2.6}$$

Question 3

$$\int_0^{2\pi} \cos(2022x) \frac{\sin(10050x)}{\sin(50x)} \frac{\sin(10251x)}{\sin(51x)} \, dx \tag{3.1}$$

Solution The **Dirichlet kernel** is defined as

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin[(n+1/2)x]}{\sin(x/2)}. \tag{3.2}$$

Therefore, we obtain

$$\begin{aligned} \frac{\sin(10050x)}{\sin(50x)} \frac{\sin(10251x)}{\sin(51x)} &= \left[1 + 2\sum_{k=1}^{100} \cos(100kx)\right]\left[1 + 2\sum_{k=1}^{100} \cos(102kx)\right] \\ &= 1 + 2\sum_{k=1}^{100} \cos(100kx) + 2\sum_{k=1}^{100} \cos(102kx) \\ &\quad + 2\sum_{k=1}^{100}\sum_{l=1}^{100} \cos[(100k + 102l)x] \\ &\quad + 2\sum_{k=1}^{100}\sum_{l=1}^{100} \cos[(100k - 102l)x]. \end{aligned} \tag{3.3}$$

According to the **orthogonality** of cosine functions, we only need to pick out the term $\cos(2022)x$ within Eq. (3.3). Since $2022 \equiv 22$ and $102 \equiv 2 \pmod{100}$, we conclude $l \equiv \pm 11 \pmod{50}$. The number pairs (k, l) with $1 \le k, l \le 100$ contributing to the term $\cos(2022)x$ are listed below:

$$(k, l) = (9, 11),\ (60, 39),\ (42, 61). \tag{3.4}$$

Finally, the integral is evaluated as

$$I = 2 \times (\pi + \pi + \pi) = 6\pi. \tag{3.5}$$

Question 4

$$\int_0^1 x^{\frac{1}{3}} (1-x)^{\frac{2}{3}} \, dx \tag{4.1}$$

Solution

$$I = B\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}{\Gamma(3)} = \frac{1}{9}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{9\sqrt{3}}. \tag{4.2}$$

Question 5

$$\left\lfloor \log_{10} \int_{2022}^{+\infty} 10^{-x^3}\, \mathrm{d}x \right\rfloor \tag{5.1}$$

Solution Consider the general integral

$$F(n) = \int_n^{+\infty} 10^{-x^3}\, \mathrm{d}x = \int_0^{+\infty} 10^{-(x+n)^3}\, \mathrm{d}x, \qquad \text{for } n > 0. \tag{5.2}$$

The upper bound can be obtained by neglecting $O(x^2)$ in the exponent

$$F(n) \le 10^{-n^3} \int_0^{+\infty} 10^{-3n^2 x}\, \mathrm{d}x = 10^{-n^3} \cdot \frac{1}{3n^2 \ln 10}. \tag{5.3}$$

The lower bound can be estimated based on the fact that the integrand is **convex**. We can refer to the triangle area under the tangent line at $x = 0$. Because we have

$$10^{-(x+n)^3} \ge 10^{-n^3} - 10^{-n^3} \cdot 3n^2 \ln 10 \cdot x = 10^{-n^3} \left(1 - 3n^2 \ln 10 \cdot x\right), \tag{5.4}$$

the lower bound is thus estimated as

$$F(n) \ge \frac{1}{2} \cdot 10^{-n^3} \cdot \frac{1}{3n^2 \ln 10}. \tag{5.5}$$

As a result, we have

$$-n^3 - \log_{10}\left(6n^2 \ln 10\right) \le \log_{10} F(n) \le -n^3 - \log_{10}\left(3n^2 \ln 10\right). \tag{5.6}$$

With $n = 2022$, we have

$$3 \times 2022^2 \times \ln 10 \approx 2 \times 10^7. \tag{5.7}$$

Hence, the original expression is evaluated as

$$I = \left\lfloor \log_{10} F(2022) \right\rfloor = -2022^3 - 8. \tag{5.8}$$

Note The definite integral over the positive real line can be calculated using the Gamma function. Note that with a **change of variable** $t = x^n$, we can obtain

$$\int_0^{+\infty} e^{-x^n}\, \mathrm{d}x = \int_0^{+\infty} \frac{t^{1/n-1}}{n} e^{-t}\, \mathrm{d}t = \frac{1}{n}\Gamma\left(\frac{1}{n}\right). \tag{5.9}$$